Appraising diversity with an ordinal notion of similarity: an Axiomatic approach

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Abstract

This paper provides an axiomatic characterization of two rules for comparing alternative sets of objects on the basis of the *diversity* that they offer. The framework considered assumes a finite universe of objects and an *a priori* given ordinal quadernary relation that compares alternative *pairs* of objects on the basis of their ordinal dissimilarity. Very few properties of this quadernary relation are assumed (beside completeness, transitivity and a very natural form of symmetry). The two rules that we characterize are the *maxi max* criterion and the *leximax* criterion. The maxi max criterion considers that a set is more diverse than another if and only if the two objects that are the most dissimilar in the former are weakly as dissimilar as the two most dissimilar objects in the later. The lexi-max criterion is defined as usual as the lexicographic extension of the maximax criterion.

1 Introduction

Would the killing of 50 000 thousand flies of a specific species have the same impact on the reduction of biological diversity than that of 200 white rhinoceros ? Is the diversity of opinions expressed in the written press larger in France than in the US ? Is the choice of models of cars offered by a particular retailer more diverse than that of another ? These are examples of questions whose answers require a precise notion of *diversity*.

Biologists have probably been the first scientists interested in developing and implementing numerical indices that aim at measuring the biological diversity offered by alternative ecosystems. One of the most widely used of these indices is Shannon (1948) *weighted entropy* measure proposed in biology by Good (1953) (see e.g. Baczkowski, Joanes and Shamia (1997),

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Baczkowski, Joanes and Shamia (1998) and Magurran (1998) for other refinements and discussions of this class of indices). The generalized Good index evaluates the diversity of any ecosystem by counting, for each species, the frequency of living individuals within the species relative to the total number of living individuals and calculates a weighted entropy over these relative frequencies. Yet, and despite its wide use and computational convenience for applications, this index lacks sound justifications. Why after all should one use the specific entropy formula for appraising the impact of major changes on biodiversity ? Answering questions like this is important in these days where many countries who have ratified the UN 1992 convention on biological diversity have adopted economically costly environmental regulations in order to prevent a deterioration of biological diversity caused by human activities. It is all the most important as the generalized entropy measure suffers from the drawback of paying no attention whatsoever to either inter-species dissimilarities, or to the possibility for two individuals of the same species to be more dissimilar than two individuals coming from different species. For instance, according to the generalized entropy formula, a world in which all living individuals are equally split between two species of fly is just as diverse as one in which the living individuals are split equally between chimpanzees and hippocampi.

Recent efforts, actually due to economists (Weitzman (1992),Weitzman (1993), Weitzman (1998)), have been made to provide axiomatic foundations to the measurement of biodiversity. Weitzman approach is based on the primitive notion of a *cardinal* numerical measure of distance between living creatures. Such a numerical distance enables one to say things such as "the biological distance between a chimpanzee and a bee is twice as large as is the biological distance between a rainbow trout and a kokanee salmon". Using such a numerical distance, Weitzman (1992) axiomatically characterizes a sophisticated iterative lexicographic method for appraising the diversity offered by a set of living individuals. Using a somewhat different setting, Bossert, Pattanaik and Xu (2002) also provide an axiomatic characterization of the Weitzman's method by taking as given a cardinal numerical measure of distances between the objects. Weitzman's procedure has been substantially generalized in a recent paper by Nehring and Puppe (2002) who propose to derive the basic numerical distance function assumed by Weitzman from an *a priori* grouping of the objects into collections of "attributes" (for instance being a mammal), each attributed being weighted by a (cardinally meaningful) numerical function.

Weitzman or Nehring and Puppe procedure, by taking due account of the (possibly) different distances that may exist between alternative pairs of living creatures, is clearly sensitive to inter-species dissimilarities. It also allows for the possibility of two individuals of a particular species (chimpanzee for instance) to be more diverse than two individuals coming from different species. On the other hand, it is not at all clear that the current state of knowledge in biology leads to such a precise cardinal measure of distance between living creatures as what is required by these approaches. All biologists would probably agree that a chimpanzee and a bee are more dissimilar than a rainbow trout and a kokanee salmon. But would they agree to say that the dissimilarity between a chimpanzee and a bee is twice that between a rainbow trout and a kokanee salmon ? Does the discriminating power of current biology enable one to perform such precise cardinal statements ?

In the last 15 years or so, interest in diversity measurement has also arisen in *non-welfarist* normative economics, in connection with the issue of comparing alternative *opportunity sets* on the basis of their freedom of choice (see e.g. Arrow (1995), Bossert (1997), Bossert (2000), Bossert, Pattanaik and Xu (1994), Dutta and Sen (1996), Foster (1993), Gravel (1994), Gravel (1998), Gravel, Laslier and Trannoy (1998), Jones and Sugden (1982), Klemisch-Ahlert (1993), Kreps (1979), Nehring and Puppe (1999), Pattanaik and Xu. (1990), Pattanaik and Xu (1998), Pattanaik and Xu (2000b), Puppe (1995), Puppe (1996), Puppe (1998), Puppe and Xu (1996), Sen (1988), Sen (1991), Sugden (1985), Suppes (1987), Suppes (1996) and Van-Hees (1997) for representative pieces of this literature and Barberà, Bossert and Pattanaik (n.d.), Foster (2001) and Sugden (1998) for surveys). A major weakness of many rankings of opportunity sets examined in this literature is their insensitivity to the diversity of the options contained in the opportunity set. Even if the fact of being forced (by lack of available alternative) to drive a blue Volkswagen Golf to go to some destination can be considered freedom-wise equivalent to being forced to make the same trip by train, this does not imply that the possibility of getting to destination by driving either a blue or a red Volkswagen Golf offers the same freedom of choice as having the possibility of making the trip either by train or by driving a red Golf. Yet many rankings of opportunity sets examined in the literature fail to make the distinction.

Albeit diversity appears to be an essential element to any definition of freedom of choice, we do not think that freedom of choice should be reduced to diversity. The precise connection between freedom of choice and diversity is a delicate issue that shall not attract our attention here. Rather, the object of this paper is to provide an axiomatic characterization of two rankings of sets on the sole basis of the diversity that they offer.

As in Weitzman (1992),Weitzman (1993), Weitzman (1998) , Bossert et al. (2002) and, to some extent, Nehring and Puppe (2002), the two rankings characterized in this paper are based on an *a priori* notion of "proximity", or "dissimilarity", between the objects that is taken as given. However, the notion of similarity on which our axioms are based requires much less information than what is necessary to define a cardinally meaningful numerical distance function such as that used in these contributions. Rather, the primitive notion of similarity on which we base our axiomatic construction is *ordinal*. That is, it requires the ability to perform statements like "the biological distance between a chimpanzee and a bee is larger than the biological distance between a rainbow trout and a kokanee salmon" but does *not* suppose the capacity of quantifying further these statements. In particular, it does not require the ability to make statements like "the biological distance between a chimpanzee and a bee is twice as large as is the biological distance between a rainbow trout and a kokanee salmon" that are required by a cardinal notion. We believe that the capacity of making cardinally meaningful statements with respect to the "similarity" of alternative pair of objects exceeds the current discriminating power of humans, even in disciplines as developed and sophisticated as biology. The very best that we can hope for in practice is a set of ordinal statements of the type: "the objects a and b are at least as dissimilar as the objects c and d." This information, that we summarize formally by a quadernary relation, is the only one used in the characterization of our two rankings of sets.

To the best of our knowledge, Pattanaik and Xu (2000a) contribution, also discussed in Bossert et al. (2002), is the only one that examines a diversity-based ranking of sets of objects that refers explicitly to an *a priori* ordinal notion of similarity. However, the ordinal notion of similarity assumed in these papers is rather crude. For it only allows objects to be either pairwise dissimilar or pairwise similar. No intermediate categories of similarities are allowed. With this "black and white" notion of similarity, Pattanaik and Xu (2000a) characterizes a ranking of sets based on the number of elements contained in the smallest (with respect to the number of elements) partition of the sets in subsets of similar objects. According to their ranking, set A is at least as diverse as set B if, and only if, the smallest partition of the set into subsets of similar objects is at least as large in A as in B. While very interesting as a first step in the process of building a diversity ranking of sets based on an ordinal notion of similarity, this result suffers obviously from the paucity of the information conveyed by the "black and white" notion of similarity used.

In this paper we characterize axiomatically two diversity rankings of sets based on an ordinal primitive notion of similarity that is not assumed to be "black and white". Rather, the primitive notion of similarity used is an abstract quadernary relation (or a binary relation on the set of all pairs of objects) that is only restricted to be reflexive, transitive and complete as well as to satisfy a weak form of symmetry. Using this notion, the first ranking that we characterize is the *maxi-max* criterion that compares sets by looking, for each set, at the pair of objects that are the most dissimilar (according to the primitive notion of similarity) and by comparing these pairs on the basis of their similarity, the more dissimilar are these maximally dissimilar object, the more diverse is the set. The other ranking characterized in this paper is the *lexi-max* criterion that is defined, as usual, as the lexicographic extension of the maxi-max criterion. It therefore compares any two sets by means of the following recursive operation. Take for each set its pair of most dissimilar objects and compare these two pairs in terms of dissimilarity. If one pair is more dissimilar than another, rank the set from which the pair is taken more diverse than the other. If the two pairs are equally dissimilar, look, in each set, for the pair of object that is ranked second in the scale of dissimilarity and compare the two pairs thus extracted. If one pair is more dissimilar than the other, rank the set from which the more dissimilar pair is extracted above the other. In case of a tie, look in each set for the third pair of objects on the scale of similarity and so on. Eventually, one either achieves a strict ranking of two pairs of objects (one pair per set) in terms of dissimilarity and ranks accordingly the corresponding sets or exhausts all possible pairs of objects that one can form in one of the two sets, in which case it considers more diverse the set that contains the most elements.

While these two rankings, and especially the second one, are of some intrinsic interest for the problem at hand, we believe that the general methodology employed for obtaining a consistent method for appraising diversity based on a primitive ordinal conception of similarity is more important than the rankings themselves. It is therefore our hope that this approach will receive more attention than it had in the literature and that our paper will contribute to this.

The rest of this paper is organized as follows. The next section presents the notation and the formal definitions of the axioms and the rankings characterized. Section 3 presents the characterization results and section 4 concludes.

2 Notations and definitions

Let X be a finite set of options (living individuals, type of means of transportations, opinions expressed in newspapers, etc.) and $\mathfrak{P}(X)$ be the set of all non empty subsets of X with generic elements A, B,.... We denote by $|A|$ the cardinality of the set A and by $P(A) = A \times A$ the set of all pairs one can form with the elements of A. Of course, $|P(A)| = |A|^2$.

At the basis of our approach is a *quadernary* relation R on X (alternatively, a binary relation on $X \times X$) (with asymmetric and symmetric factors P and I respectively) which reflects *ordinal* knowledge about the dissimilarity that exists between options. In this light, the statement $(w,\ z)\ R\ (x,\ y)$ is interpreted as meaning "the objects w and z are at least as dissimilar as the objects x and y". To motivate this interpretation, we assume throughout that $(x,\ y)\ R\ (x, x)$ for every distinct objects x and y and that (x, x) $I\ (y, y)$ (that is, two distinct objects are always weakly more dissimilar than are one of the two objects and itself, and pairs of identical objects are just equally similar). These two properties would clearly hold true if, as Weitzman (1992) or Bossert et al. (2002), we would accept to go as far as measuring the dissimilarity by a (cardinally measurable) distance function $d : X \times X \to \mathbb{R}$. We assume further that R is *symmetric* in the sense that (x, y) R (y, x) holds for every objects x and y, is *complete* (i.e. either $(x,\ y)$ R $(w,\ z)$ or $(w,\ z)$ R $(x,\ y)$ holds for every (not necessarily distinct) x, w, y, $z \in X$ and is *transitive* (i.e. for every (not necessarily distinct) objects u, v, x, w, y, $z \in X$, $(x,\ y)$ R $(w,\ z)$ and $(w,\ z)$ R $(u,\ v)$ must always imply $(x,\ y)$ R $(u,\ v)$. The reader can easily verify that these properties are satisfied by the ranking of pairs of objects induced by conventional distance function d (in particular d is conventionally assumed to be symmetric). In order to simplify some of the proofs, we further assume that R is such that $(x,\ y)$ P (x, x) for every two distinct x and y (two distinct options are always strictly more dissimilar than one of the two options and itself). We let $\mathfrak{R}$ denote the set of all ordinal quadernary relations that satisfy all these properties. We record the obvious following fact (whose obvious proof is omitted).

Fact 1 *If R is a dissimilar quaternary relation in $\mathfrak{R}$, then, for all distinct x and $y \in X$, and for all $z \in X$, (x,y) P (z,z)*

Using R, one can order, for any set $A \in \mathfrak{P}(X)$, the pairs in $P(A)$ to form the ordered $|A|^2$-dimensional vector of pairs $Z(A)$ defined as follows:

$Z(A) = (a_{(1)},\ a_{(2)},\ ...,\ a_{(|P(A)|)})$

where $a_{(1)} = (x, y)$ for some x and y in A denotes the "greatest" (or most dissimilar) pair according to R

$a_{(2)}$ denote the second "greatest" pair according to R,

...

$a_{(|P(A)|)}$ denote the "smallest" pair according to R.

Remark 1 *As R is symmetric, there is some arbitrariness in constructing the vector $Z(A)$ (the order of appearance of any two symmetric pairs (x, y) and (y, x) is obviously irrelevant). In the same vein, we also know, thanks to fact 1, that the $|A|$ last pairs of the vector $Z(A)$ are precisely the duplications of the $|A|$ elements of A.*

We let $\succeq$ (with asymmetric and symmetric factors $\succ$ and $\sim$ respectively) be a *transitive* binary relation on $\mathfrak{P}(X)$ that aims at reflecting the evaluation of the diversity offered by alternatives sets of objects in $\mathfrak{P}(X)$. We interpret the statement $A \succeq B$ as meaning "the set A offers at least as much diversity as the set B"

We wish to propose plausible properties (axioms) that $\succeq$ could satisfy to serve as a sensible method for appraising diversity, taking as given the ordinal conception of dissimilarity reflected in R. In order to formulate these, we define, for any set of objects $A \subseteq X$, and for any pair of objects x, $y \in X$, the set $P_A(x, y) \subseteq A \times A$ of all pairs of objects in A that are at least as dissimilar as the objects x and y. This definition is as follows.

Definition 1 $P_A(x,y) = \{(a,a') \in A \times A : \quad (a,a')\ R\ (x,y)\}$

Without loss of generality, we write the pairs of $P_A(x,y)$ in decreasing order of dissimilarity so that $P_A(x,y) = \{a_{(1)}, ..., a_{\ |(P_A(x,y)|}\}$ (where $a_{(i)} \in A \times A$ for all $i = 1, ..., |(P_A(x,y)|$ with $a_{(j)}\ R\ a_{(j+1)}$ for all $j = 1, ..., |(P_A(x,y)| - 1\}$. We now present the axioms used in the characterizations by what seems in our view to be their decreasing order of intuitive plausibility.

Axiom 1 $\forall\ w, x,\ y,\ z\ \in X,\quad (w,\ z)\ R\ (x,\ y) \Longleftrightarrow \{w,\ z\} \succeq \{x,\ y\}$

Axiom 2 $\forall\ A,\ B \in P(X),\ \ if\ \ A \supseteq B,\ then\ A \succeq B..$

Axiom 3 $\forall\ A,\ B, C\ and\ D \in \mathfrak{P}(X)\ such\ that\ B\cap C = B \cap D = C\cap D = \emptyset,$ $(A \succsim B\cup C,\ A \succsim B\cup D\ and\ A \succsim C\cup D) \Longrightarrow A \succsim B\cup C\cup D\ and\ (A \succ (B\cup C),$ $A \succ (B \cup D)\ and\ A \succ (C \cup D)) \Longrightarrow A \succ (B \cup C \cup D)$

Axiom 4 $\forall\ A,\ B \in \mathfrak{P}(X)\ such\ that\ |A| = |B|\ a_{(i)}\ R\ b_{(i)}\ \forall\ i\ with\ a(i) \in Z(A)\ and\ b(i) \in Z(B) \Longrightarrow A \succeq B$

Axiom 5 *For all* $w, x, y, z \in X,\ if\ \{w, z\} \succ \{x, y\}\ and\ C, D\ \in \mathfrak{P}(X)$ *are such that*
$\left|P_{C\cup\{w,z\}}(w,z)\right| = \left|P_{D\cup\{x,y\}}(x,y)\right|$ *and* $a_{(i)}\ R\ b_{(i)}\ for\ a_{(i)} \in P_{C\cup\{w,z\}}(w,z)$ *and* $b(i) \in P_{D\cup\{x,y\}}(x,y)$ *for all* $i \in \{1, ..., \left|P_{C\cup\{w,z\}}(w,z)\right|\}$ *then* $C \cup \{w,z\} \succ B \cup \{x,y\}$.

Axiom 1 just says that the ranking of pairs in terms of diversity must *coincide* with the ranking of the pairs in terms of dissimilarity as *per* the quadernary relation R. It is difficult to imagine a diversity-ranking of sets based on an *a priori* notion of dissimilarity between options that would violate this axiom. Axiom 2 is the widely discussed (at least in the freedom of choice literature mentioned in introduction) axiom of *weak monotonicity with respect to set inclusion.* It is also a hardly disputable axiom in the context of diversity measurement. For it is quite difficult to imagine a plausible conception of diversity that would consider that *adding* an object to a set could strictly reduce its diversity. Axiom 3 is certainly more disputable than the two preceding ones but is not unreasonable. Consider two sets A and B such that A is considered more diverse (weakly or strictly) than B. Consider then two processes of adding a bunch of options to B. One process consists in adding to B options collected into a set C while the other process consists in adding to B options gathered into some other set D (disjoint from C). Assume that the enlargement of diversity offered by B as a result of either of these two process is insufficient to reverse the relative ranking of the enlarged set with respect to A. In a situation like this, the axiom requires that, if A offers weakly or strictly more diversity than that provided by all options added to B by the two processes, then A should also be considered (weakly or strictly) more diverse than B enlarged by all objects in the two sets C and D. To give a somewhat more intuitive example, assume that A consists in all currently living creatures categorized as mammals, B contains all cartilagineous fishes, C contains all osseous fishes and D consists in all batracians. Axiom 3 would then require that if the diversity offered by mammals is larger than that offered by all fishes (cartilaginous and osseous), is larger than that offered by cartilagineous fishes and batracians and is also larger than that offered by all batracians and osseous fish, then the set of all mammals should also be considered more diverse than the set of all fishes and batracians.

Axiom 4 connects, for sets that contain the same number of options, the judgements made with respect to the dissimilarity of their options and those with respect to their relative standing in terms of diversity. As sets with the same number of options have obviously the same number of pairs, the ladder of pairs from the most dissimilar to the least dissimilar in the sets Z defined above will obviously have the same number of steps. Axiom 4 requires in such a case that if, for every step, the corresponding pair is weakly more dissimilar in A than in B, then A should be considered weakly more diverse than B.

Axiom 5 is, probably, the more difficult to accept (and to understand). It says that if a pairs of objects is considered strictly more diverse (or dissimilar if axiom 1 is assumed) than another, then this ranking should be robust to a certain form of addition of options in each set. The restriction imposed on the addition of options is that all newly created pairs of options that are weakly more dissimilar, in each set, to the initial pair should be related, pair by pair, by a dominance relation with respect to dissimilarity. Here again, an example may help. Assume that the diversity ranking of sets considers the pair {*bee*,*chimpanzee*} to be strictly more diverse than the pair {*kokanee salmon*, *rainbow trout*}. Suppose we add a fly to the first set and a brown trout to the second set and assume that the following (plausible) dissimilarity statements hold with respect to the living individuals:

1) the chimpanzee is weakly more similar to the bee than to the fly,

2) the bee is weakly more similar to the fly than to the chimpanzee,

3) a kokanee salmon is weakly more dissimilar to the brown trout than it is to the rainbow trout

4) brown and rainbow trouts are more similar than the rainbow trout and the kokanee salmon

5) the chimpanzee and the fly are more dissimilar than the kokanee salmon and the brown trout.

We note that, in this example, there is only one pair of objects that are weakly more diverse than the original pair in each set. These pairs are (*chimpanzee*, *fly*) for the first set and (*brown trout*, *kokanee salmon*) for the second. We note also that the first of these pairs is more diverse than the second one. Axiom 5 would then require the set $\{$*bee*, *chimpanzee*, *fly*$\}$to be strictly more diverse than the set $\{$*brown trout*, *kokanee salmon*, *rainbow trout*$\}$. Although not as transparent as one would like, axiom 5 is not unreasonable. It roughly requires the ranking of two pairs in terms of diversity to remain if an increase in diversity is, in a precise sense, larger in the dominating set than in the dominated one.

We now formally define the two specific rankings of sets that we characterize by these axioms in the next section. The first of these rankings is the *maxi-max* criterion that ranks sets according to the relative dissimilarities of their most dissimilar pairs. This ranking $\succsim_{\max}$is defined as follows.

Definition 2 *For all* A, $B \in \mathfrak{P}(X)$, $A \succeq_{\max} B \Longleftrightarrow a_{(1)}\ R\ b_{(1)}$

To illustrate, suppose that X is the set of all means of transportation available to perform a certain trip between two cities defined specifically as $X = \{train,\ volkswagen,\ lada, bike,\ foot\}$. Assume also that the ordinal notion of dissimilarities between these means of transportations is given by the quadernary relation R defined by (*train*,*foot*) I (*volkswagen*,*foot*) P (*lada*,*foot*) P (*train*,*bike*) I (*volkswagen*,*bike*) P (*lada*, *bike*) P (*train*, *volkswagen*) P (*train*, *lada*) P (*bike*,*foot*) P (*volkswagen*,*lada*). Then the maxi-max criterion would consider that the set $\{train, foot\}$ offers just as much diversity as the set $\{train,\ volkswagen,\ lada, bike,\ foot\}$, a judgement which may sounds at odd with one's intuition of what is diversity. Its biggest weakness is obviously that it only focus on the two most dissimilar objects in the sets and it ignores completely the contribution to diversity made by the presence of less dissimilar objects. The lexi-max criterion $\succsim_{lex}$defined as follows avoids to some extent this weakness.

Definition 3 *For all* A, $B \in \mathfrak{P}(X)$, $A \succ_{lex} B \Longleftrightarrow$ *either there exists some* $k \in \{1, ..., \min(|A|^2, |B|^2)\}$ *such that* $a_{(k)}\ P\ b_{(k)}$ *and* $a_{(i)}\ I\ b_{(i)}$ *for* $i = 1, ...k-1$ *or* $|A| > |B|$ *and* $a_{(i)}\ I\ b_{(i)}$ *for* $i = 1, ..., |B|^2$ *and* $A \sim_{lex} B \Longleftrightarrow |A| = |B|$ *and* $a_{(i)}\ I\ b_{(i)}$ *for* $i = i = 1, ..., |B|^2$

Albeit this ranking expresses some sensitivity with respect to the contributions of options that are not maximally dissimilar to diversity (for instance by considering that the set $\{train,\ volkswagen,\ lada, bike,\ foot\}$ is strictly more diverse than the pair $\{train, foot\}$), this sensitivity is not as great as one would like. For it nonetheless gives a "veto power" to the most dissimilar two objects in the sets with respect to the appraisal of their diversity. In the example above, the set $\{train, foot\}$ would be considered strictly more diverse than the set $\{train,\ lada,\ bike\}$ as per the criterion $\succsim_{lex}$even though this judgement may hurt the intuition of someone who nonetheless accepts the notion of dissimilarity R assumed in this example.

It would therefore be nice to have a diversity ranking that enables more trade-off between the contributions of alternative pairs of options to diversity than what is allowed by the two diversity orderings characterized in this paper.

We now turn to this characterization.

3 Characterization Results

We first provide the characterization of $\succeq_{\max}$by means of axioms 1-3.

Theorem 1 *Let $\succeq$ be transitive binary relation defined on $\mathfrak{P}(X)$ and let R be an ordinal notion of similarity belonging to $\mathfrak{R}$. Then $\succeq$ satisfies Axioms 1 to 3 if and only if $\succeq \ = \ \succeq_{\max}$*

Proof. Necessity. It is immediate to see that the transitive binary relation $\succeq_{\max}$satisfies axioms 1 and 2. As for axiom 3, suppose that $A \succeq_{\max} B \cup C = E$, $A \succeq_{\max} B \cup D = F$ and $A \succeq_{\max} C \cup D = G$ and let $H = B \cup C \cup D$. Then $a_{(1)}\ R\ e_{(1)}$, $a_{(1)}\ R\ f_{(1)}$ and $a_{(1)}\ R\ g_{(1)}$ for $a_{(1)}$, $e_{(1)}$, $f_{(1)}$ and $g_{(1)}$ denoting respectively the first component of the vectors $Z(A)$, $Z(E)$, $Z(F)$ and $Z(G)$. We therefore have $a_{(1)}\ R\ \max_R(e_{(1)}, f_{(1)}, g_{(1)})$ which clearly implies that $A \succeq_{\max} H$.

Sufficiency We show first that if $\succeq$ is transitive and satisfies axioms 1 to 3, then we have, for every A and $B \in \mathfrak{P}(X)$, $A \succeq B \Longrightarrow A \succeq_{\max} B$. Suppose $A \succeq B$. By axiom 2, we have $B \succeq \{b_{(1)}\}$ and by transitivity, $A \succeq \{b_{(1)}\}$. If $|A| = 2$, $A = \{a_{(1)}\}$, so $\{a_{(1)}\} \succeq \{b_{(1)}\}$ and by axiom 1, $a_{(1)}\ R\ b_{(1)}$, which means $A \succeq_{\max} B$. If $|A| > 2$, write $A = \{a_1, ..., a_{|A|}\}$ and assume by contradiction that $a_{(1)}\ R\ b_{(1)}$ is false. Since R is complete, this amounts to assuming that $b_{(1)}\ P\ a_{(1)}$ and, therefore, that $b_{(1)}\ P\ (a_i, a_j)$ for all i, $j \in \{1, ..., |A|\}$. Pick up any option a_1 in A and let $b_{(1)} = (b_1, b_2)$. One thus has, by axiom 1, that $\{b_1, b_2\} \succ \{a_1, a_i\}$, $\{b_1, b_2\} \succ \{a_1, a_j\}$ and $\{b_1, b_2\} \succ \{a_i, a_j\}$ for all i, $j \in \{1, ..., |A|\}$. By axiom 3, we must have $\{b_1, b_2\} \succ \{a_1, a_i, a_j\}$. Redoing the same procedure while replacing the option a_j by some option $a_h \in A$, one obtains that$\{b_1, b_2\} \succ \{a_1, a_i, a_h\}$. Using axiom 3 again and the fact that $\{b_1, b_2\} \succ \{a_j, a_h\}$, one is led to the conclusion that $\{b_1, b_2\} \succ \{a_1, a_h, a_i, a_j\}$. Redoing the last procedure if necessary while replacing a_h by $a_g \in A$, one can analogously obtain the statement $\{b_1, b_2\} \succ \{a_1, a_g, a_i, a_j\}$ and combining the last two statements and the fact that $\{b_1, b_2\} \succ \{a_g, a_h\}$, one obtains again by axiom 3 that $\{b_1, b_2\} \succ \{a_1, a_g, a_h, a_i, a_j\}$. This procedure can clearly be repeated with as many options in A is needed to finally obtain the required contradictory conclusion that $B \succeq \{b_1, b_2\} \succ A$. We now show that if $\succeq$ is transitive satisfies axioms 1 to 3, then we have $A \succeq_{\max} B \Longrightarrow A \succeq B$. Suppose $A \succeq_{\max} B$. Then $a_{(1)}\ R\ b_{(1)}$ where $a_{(1)}$ and $b_{(1)}$ are the first components of the vectors $Z(A)$ and $Z(B)$ respectively..Let $|B| = m$, $B = \{b_1, b_2, ..., b_m\}$, $a_{(1)} = (a_1, a_2)$ and $b_{(1)} = (b_1, b_2)$. If $m = 2$, then $\{a_1, a_2\} \succeq \{b_1, b_2\}$ and, by axiom 2, $A \succeq \{a_{(1)}\}$, so that $A \succeq B$. For more general cases, we show the result by induction. For that purpose, we start with the case $m = 3$. Because $a_{(1)}\ R\ b_{(1)}$, we have $a_{(1)}\ R\ (b_1, b_2)$, $a_{(1)}\ R\ (b_1, b_3)$ and $a_{(1)}\ R\ (b_2, b_3)$. Using axiom 1, we can write $\{a_1, a_2\} \succeq \{b_1\} \cup \{b_2\}$, $\{a_1, a_2\} \succeq \{b_1\} \cup \{b_3\}$ and $\{a_1, a_2\} \succeq \{b_2\} \cup \{b_3\}$. By axiom 3, it follows that $\{a_1, a_2\} \succeq \{b_1\} \cup \{b_2\} \cup \{b_3\}$ and by axiom 2, that $A \succeq \{b_1, b_2, b_3\} = B$. The case $m = 3$ is then proved. Now suppose the result is true for any $m \in \{3, ..., |X| - 1\}$. That is, suppose that if A is a set in $\mathfrak{P}(X)$ and B is another set in $\mathfrak{P}(X)$ such that $|B| = m$, then $A \succeq_{\max} B \Longrightarrow A \succeq B$ and suppose $A \succeq_{\max} B'$ where $B' = B \cup \{b_{m+1}\}$. Let $\overline{B} = \{b_1, b_2, ..., b_{m-1}\}$, $C = \{b_m\}$ and $D = \{b_{m+1}\}$. By hypothesis, $A \succeq \overline{B} \cup C$. Because $a_{(1)}\ R\ b'_{(1)}$, we have $a_{(1)}\ R\ (b_m, b_{m+1})$, so $A \succeq C \cup D$. Finally, let $B'' = \overline{B} \cup D$. Then $b'_{(1)}\ R\ b''_{(1)}$ and $a_{(1)}\ R\ b''_{(1)}$. We therefore have that $A \succeq_{\max} B''$. Yet $|B''| = m$ so that, by the induction hypothesis, we have $A \succeq B''$. By axiom 3, we have $A \succeq \overline{B} \cup C$, $A \succeq \overline{B} \cup D$ and $A \succeq C \cup D$, so that $A \succeq \overline{B} \cup C \cup D$, and this concludes the proof. ■

We first remark that, albeit this was not required, we obtain the completeness of the ranking as a by-product of the three axioms. It is also worth noticing that the characterization of $\succeq_{\max}$is obtained from the (reasonably) intuitive axioms 1 to 3 that only uses properties of sets. Only axiom 1 makes the connection between the ranking of specific sets - the pairs - and the underlying notion of dissimilarity R. Unfortunately the characterization of the more interesting $\succeq_{lex}$criterion is obtained from the axioms 1,4 and 5, all of which are explicitly based on the properties of the pairs of the sets.

Before turning to this characterization, we show that axioms 1 to 3 used to characterize $\succeq_{\max}$are independent.

Proposition 1 *Axioms 1 to 3 are independent.*

Proof. Let $\succeq_*$be defined by: $A \succeq_* B \Longleftrightarrow b_{(|P(B)|-|B|)}\ R\ a_{(|P(A)|-|A|)}$. This transitive and complete binary relation on $\mathfrak{P}(X)$ considers that A offers at least as much diversity as B if and only if the two most similar distinct objects in B are weakly more dissimilar than the most similar distinct objects in A. It is certainly a peculiar criterion for comparing sets on the basis of their diversity. It is immediate to see that $\succeq_*$violates axiom 1 and that it satisfies axiom 2. To see that $\succeq_*$satisfies axiom 3, assume that $A \succeq_* B \cup C$, $A \succeq_* B \cup D$ and $A \succeq_* C \cup D$. Then one has $b_{(|P(B\cup C)|-|B\cup C|)}\ R\ a_{(|P(A)|-|A|)}$, $b_{(|P(B\cup D)|-|B\cup D|)}\ R\ a_{(|P(A)|-|A|)}$ and $b_{(|P(C\cup D)|-|C\cup D|)}\ R\ a_{(|P(A)|-|A|)}$. Clearly, since either

$$\begin{aligned} b_{(|P(B\cup C\cup D)|-|B\cup C\cup D|)} &= b_{(|P(B\cup C)|-|B\cup C|)}, \\ b_{(|P(B\cup C\cup D)|-|B\cup C\cup D|)} &= b_{(|P(B\cup D)|-|B\cup D|)} \text{ or} \\ b_{(|P(B\cup C\cup D)|-|B\cup C\cup D|)} &= b_{(|P(C\cup D)|-|C\cup D|)} \end{aligned}$$

one has

$$b_{(|P(B\cup C\cup D)|-|B\cup C\cup D|)}\ R\ a_{(|P(A)|-|A|)}$$

and, therefore, $A \succeq_* B \cup C \cup D$. Now let $\succeq_D$be defined by $A \succeq_D B \Longleftrightarrow a_{(|P(A)|-|A|)} \; R \; b_{(1)}$. This ordering says that A offers at least as much diversity as B if and only if the two most similar distinct object in A are at least as dissimilar as any pairs of objects in B. It is immediate to see that $\succeq_D$ satisfies axioms 1. To see that it verifies axiom 3, assume that $A \succeq_D (B \cup C) = E$, $A \succeq_D (B \cup D) = F$ and $A \succeq_D C \cup D = G$. Then $a_{(|P(A)|-|A|)} \; R \; e_{(1)}$, $a_{(|P(A)|-|A|)} \; R \; f_{(1)}$ and $a_{(|P(A)|-|A|)} \; R \; g_{(1)}$ and, therefore, $a_{(|P(A)|-|A|)} \; R \; h_{(1)}$ where $H = B \cup C \cup D$ and $e_{(1)}$, $f_{(1)}$ $g_{(1)}$ and $h_{(1)}$ denote, respectively, the first components of the vectors $Z(E)$, $Z(F)$, $Z(G)$ and $Z(H)$. To see that $\succeq_D$violates axiom 2, let $A = \{a_1, \ a_2\}$ and $B = \{a_1, \ a_2, \ a_3\}$ and assume that R is such that $(a_1, \ a_2) \; P \; (a_1, \ a_3) \; R \; (a_2, \ a_3)$. Clearly, $a_{(1)} = (a_1, \ a_2)$, $b_{(|P(B)|-|B|)} = (a_2, a_3)$ and, therefore $B \not\preceq A$. Finally, let $\succeq_{add}$be defined by:$A \succeq_{add} B \Longleftrightarrow \sum_{i=1}^{|A|^2} v(a_{(i)}) \geq \sum_{i=1}^{|B|^2} v(b_{(i)})$ for some function $v \in X \times X \to \mathbb{R}_+$ such that, for all (w, z), $(x, y) \in X \times X$, $v(w, z) \geq v(x, y) \Leftrightarrow (w, z) \; R \; (x, y)$. Such an ordering clearly satisfies axioms 1 and 2. Yet, it may violates axiom 3 if, for instance, $X = \{w, x, y, z\}$ and v is such that $v(w, z) = 7$, $v(w, y) = 5$, $v(w, x) = 3 = v(x, y)$. In such a case, defining $A = \{w, z\}$, $B = \{w\}$, $C = \{y\}$ and $D = \{x\}$, one has $A \succeq_{add} B \cup C \Leftrightarrow 7 \geq 5$, $A \succeq_{add} B \cup D \Leftrightarrow 7 \geq 3$ and $A \succeq_{add} C \cup D \Leftrightarrow 7 \geq 3$. Yet $A \prec_{add} B \cup C \cup D$ as $v(w, z) = 7 < v(w, y) + v(w, x) + v(x, y) = 11$. ■

We now turn to the axiomatic characterization of $\succeq_{lex}$.

Theorem 2 *Let $\succeq$ be a transitive binary relation on $\mathfrak{P}(X)$ and let R be an ordinal notion of similarity belonging to $\mathfrak{R}$. Then $\succeq$ satisfies Axioms 1, 4 and 5 if and only if $\succeq \; = \; \succeq_{lex}$*

Proof. It is immediate to verify that $\succeq_{lex}$, a transitive and complete binary relation on $\mathfrak{P}(X)$, satisfies axioms 1, 4 and 5. Assume now that $A \succ_{lex} B$. Then, by definition, either there exists a $k \in \{1, ..., \min(|A|^2, |B|^2)\}$ such that $a_{(k)} \; P \; b_{(k)}$ and $a_{(i)} \; I \; b_{(i)} \; for \; i = 1, ...k-1$ or $|A| > |B|$ and $a_{(i)} \; I \; b_{(i)} \; for \; i = 1, ..., |B|^2$. We can not be in the later case because, as noticed above (remark 1), the $|A|$ last pairs of the vector $Z(A)$ are the duplications of the $|A|$ elements of A while the $|B|$ last pairs of the vector $Z(B)$ are the duplications of the $|B|$ elements of B. As $|A| > |B|$, and $(x, y) \; P \; (z, z)$ for every $x, y, z \in X$ (with $x \neq z$), this implies the existence of a component k of the vector $Z(A)$ and $Z(B)$ such that $a_{(k)} \; P \; b_{(k)}$. Assume therefore that we are in the case where there exists a $k \in \{1, ..., \min(|A|^2, |B|^2)\}$ such that $a_{(k)} \; P \; b_{(k)}$ and $a_{(i)} \; I \; b_{(i)} \; for \; i = 1, ...k-1$. Writing $a_{(k)} = (a_0^k, a_1^k)$ and $b_{(k)} = (b_0^k, b_1^k)$ (with $a_j^k \in A$ and $b_j^k \in B$ for every $k \in \{1, ..., \min(|A|^2, |B|^2)\}$, $j = 0, 1$) we have by axiom 1 that $\{a_0^k, a_1^k\} \succ \{b_0^k, b_1^k\}$. Let $C = A \backslash \{a_0^k, a_1^k\}$ and $D = B \setminus \{b_0^k, b_1^k\}$. C and D are clearly like the sets of axiom 5. In particular, one has $\left| P_{\{a_0^k, a_1^k\} \cup C}(a_{(k)}) \right| =$

$\left|P_{\{b_0^k,b_1^k\}\cup D}(b_{(k)})\right|$ and $a_{(i)}\ I\ b_{(i)}\ if\ a_{(i)} \in P_{\{a_0^k,a_1^k\}\cup C}(a_{(k)})$ and $b_{(i)} \in P_{\{b_0^k,b_1^k\}\cup D}(b_{(k)})$. By application of axiom 5, we obtain $A \succ B$.Suppose now $A \sim_{lex} B$. Then $a_{(i)}\ R\ b_{(i)}\ \forall\ i$ and $b_{(i)}\ R\ a_{(i)}\ \forall\ i$ and $|A| = |B|$.By axiom 4, we then have $A \succeq B$ and $B \succeq A$ which entails $A \sim B$. Now, we have to show that if $\succeq$ is transitive and satisfies axioms 1, 4 and 5, then we have, for every A and $B \in P(X)$, $A \succeq B \Longrightarrow A \succeq_{lex} B$. Assume by contradiction that the implication is false. Then, since $\succeq_{lex}$is complete, this amounts to say that $B \succ_{lex} A$ holds which, by virtue of what we just established, implies that $B \succ A$, a contradiction. ■

Although a characterization of $\succsim_{lex}$which would use axioms that refer to *elements* of the sets rather than to *pairs of elements* would be more elegant, such an axiomatization is not easy to obtain. The leximax criterion is clearly an ordering in which the precise positions occupied by the pairs in the dissimilarity scale matters a great deal. Yet, it is difficult to axiomatically control these positions by simply using properties of the single elements. For instance, adding elements in the way suggested by axiom 3 does not do because the whole ranking of pairs in the set is severely affected by these additions.

We conclude this section by proving that axioms 1, 4 and 5 are indepedent.

Proposition 2 *Axioms 1, 4 and 5 are independent.*

Proof. Consider first $\succeq_{add}$as defined in the proof of proposition 1. It satisfies as we have seen axiom 1 and it is not hard to see that it also satisfies axiom 4. To see that it violates axiom 5, consider $X = \{w, x, y, z\}$ and assume that the function v which defines $\succeq_{add}$is such that $v(w, z) = 7$, $v(w, y) = 5$, $v(w, x) = 3$, $2 = v(x, y) = v(x, z)$ and $v(y, z) = 0$. In such a case, we have that $\{w, y\} \succ \{w, x\}$ and $C = \{z\} = D$ are just as in the antecedent of axiom 5 (with $P_{\{w,y,z\}}(w, y) = \{(w, z), (w, y)\}$ and $P_{\{w,x,z\}}(w, x) = \{(w, z), (w, x\})$. Yet $\{w, y, z\} \preceq \{w, x, z\}$ since $7 + 5 + 0 \leq 7 + 3 + 2$ in contradiction of the consequence of axiom 5. Consider now the widely discussed cardinality ordering $\succsim_{CARD}$defined by $A \succsim_{CARD} B \Leftrightarrow |A| \geq |B|$. This ordering obviously violates axiom 1 for any dissimilarity notion contained in $\mathfrak{R}$. It satisfies however axiom 4 and, trivially, axiom 5 (whose antecedent never applies when there is universal indifference between all pairs of objects. Consider finally the incomplete transitive binary relation $\widehat{\succeq}_{lex}$ defined by $A\ \widehat{\succeq}_{lex}\ B \Longleftrightarrow A \succ_{lex} B$. As can be seen in the proof of theorem 2, this binary relation satisfies axioms 1 and 5. Yet it fails to satisfy axiom 4 (since $\widehat{\succeq}_{lex}$ considers as non comparable any two sets, such as those mentioned in axiom 3, that would be considered indifferent by $\succeq_{lex}$. ■

4 Conclusion

The purpose of this paper was to investigate the possibility of deriving axiomatic ranking of sets of objects on the basis of their diversity by using only an ordinal primitive information about the similarities of the objects. This approach is to be contrasted with the most recent sophisticated ones such as those proposed by Weitzman (1992),Weitzman (1993), Weitzman (1998), Nehring and Puppe (2002) or Bossert et al. (2002) which assume a cardinally measurable primitive notion of similarities. While this investigation has been proved successful, we are aware that the specific rankings which we have characterize in this paper are far from perfect. As mentioned earlier, a basic flaw with these two rankings is that they do not allow smooth trade off between the contributions of alternative pairs of objects to diversity. Both rankings give a very large "veto power" to the two most dissimilar options in the sets to compare the relative diversity that they offer. It would be nice to obtain "smoother" rankings of sets than the two characterized in this paper. An interesting class of these rankings would be an additive one, a typical member of which would view the diversity of as set as the sum of values assigned to each of its pairs by a function that numerically represent, in the sense of Debreu (1954), the binary relation R defined on $X \times X$. An example of such a ranking is the ordering $\succeq_{add}$considered in the proof of proposition 1. Finding an axiomatic characterization of such a family of diversity ranking is worthwhile objective for further research. Another one is to try to relate the measurement of diversity with that of freedom of choice.

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