Asset Pricing Bubble Formation with Heterogenous Agents

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Abstract

In this paper we study the accruement and decay of asset pricing bubbles under the assumption that young agents behave boundedly rational when first entering the market and then gain more and more experience when growing older, finally reaching a state of perfect rational behavior. Therefore we set up an overlapping generations model where agents form their beliefs about the payoff of a risky asset by compiling a rational signal element and an irrational momentum component. Young generations of traders are unexperienced in fundamental signal interpretation and thus put high weight on momentum observation whereas old generations overweight the signal term when making their decisions. We show that bubbles can arise and deflate endogenously in this setting when a signal arrives, and that experienced traders can make profit in the market whereas unexperienced traders will loose money. Overall lifetime performance is dependent on the phase that the market is in when a trader first enters it. Also we analyze exploiting behavior of experienced traders and show that after a shock they tend to push prices beyond the myopic optimum to induce strong momentum in the following period when optimizing their intertemporal utility in a dynamic way.

JEL-Classification: ***G12, D83***

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1 Introduction

There is a couple of models that try to model asset pricing behavior in the absence of perfectly rational behavior. Psychological research supports this

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approach as it shows that human beings are by no means able to estimate and calculate all possible developments in asset pricing determinants and that even trained economists who theoretical have a pretty accurate idea of the mechanisms working at financial markets are far from avoiding crucial mistakes when facing real world investment decisions. The same holds true for the practical trained group of professional traders who often even underperform market benchmarks as shown for example in **?** (**?**). Nevertheless no one would doubt that frequent experience in trading should improve ones ability to infer correct conclusions when observing signals from the environment. This is not an exceptional idea, as a characteristic feature of human learning is that it is fostered best by gaining experience and learning from errors when repeating a similar task again and again. All the more amazing it is that economic theory so often ignores the process of learning rational behavior in complex environments *itself* and instead assumes that all economic subjects already finished this process and as a result behave in a perfectly rational way. Of course this assumption makes sense when considering a relatively simple task where the learning period is comparatively short in relation to the period of action. For example, you very quickly learn not to touch a hot plate once you tried it and then behave rational in this respect for the rest of your life. Also, there are many problems where solutions can be found in the total absence of any own experience solely by logical deduction - but mostly this fields are deterministic in nature or at least of a low degree of complexity. Matters are different when considering complex tasks where learning accounts for long time spans and even continues while the task is already exercised. An example would be scientific research. Even while executing his job, an economist for example will hopefully constantly learn how to do his job even better and thus his research should improve over time. In a sense his behavior will get more and more optimal in a constant process of learning, thus learning improves his *ability* to "behave" optimal.

In contrast learning in economic settings mostly deals with the acquirement of new information and not the improvement of ones capability of correct information processing respectively correct decision making. We claim that exactly this process of decision enhancement might play an important role in a large set of complex economic phenomena and that perfect rational behavior in some environments is - if at all - only reached at the end of more or less long development process. As we see it, estimating asset values is part of this set as it is influenced by so many exogenous variables and psychological motives of other traders that it seems doubtful that something like a "perfectly rational decision" could ever be achieved. At least it will take a long period to understand all underlying forces necessary to make perfectly rational decision in the end. Another example to underpin our sceptical view is the "information efficiency" defined by Fama (**?**). It claims that all available information will immediately be incorporated into prices on a financial market. However all information means *all* information, that is prices will even reflect the fact that a trader thousand kilometers away needs some cash to buy a new car because his old one broke down the last day and therefore sells a specific stock on the market. Concerning the man this is a rational decision to rebalance his utility, but in aggregation all this information will leads to an overflow in parameters and it seems pretty unrealistic that it is possible to compute all this eventualities in the real (and non static) world in finite time and even more unrealistic to do so before even newer information arrives. The problem therefore is not too less but *too much* information and markets are possibly in a way too efficient in incorporating information to let all traders keep the pace. There is a wide literature that indicates that human beings use heuristics to keep up with this overflow of information. It seems plausible that these heuristics will improve over time, and in our model we will allow them to finally reach a state where perfectly rational decisions are made. We hope that focusing on the evolution of increasingly efficient decision heuristics might add to the understanding of some of the puzzles raised by the assumption that agents instantly behave in a perfectly rational way.

The remainder of the paper is organized as follows:
After another excursus on motivation, this chapter also will display a survey of related literature. In section 2 we will introduce our model and state propositions for the basic setting. In section 3 we will present some results of numerical simulations as price paths and life cycle performance of traders. In section 4 we introduce strategic behavior and dynamic optimization of rational agents in our model. We present the solutions of these problems in section 5 and section 6 concludes.

1.1 Motivation

The motivation for this paper is twofold. First we have an explanatory goal as to explain the phenomena of underpricing, overpricing, excess volatility, life cycle trading returns and the behavior of stock prices after shocks to fundamental values.
Second we have a methodological intention as we want to introduce a new aspect to economic modelling by considering the improvement in personal decision strategies over time in an economic model. To the best of our knowledge this aspect has not been raised before and we think that exploring it could help to understand some puzzling patterns in observed human behavior. In particular we will try to shed light on how these heterogenous and improving heuristics might influence the development of supply and demand and hence prices in an asset market.

We think that our approach might be helpful to better understand the formation of trading strategies and its impact on market prices, but nevertheless we know that it is always risky to deviate from classical theory as one quickly enters a somewhat dusty area of scientific uncertainty. However, we are convinced that it is nevertheless necessary to explore that region in the hope of finding some new and possibly useful supplementary insights within.

In any case, the departure from widely accepted assumptions should only be done with extreme care and only if no other way seems to be passable. Hong and Stein (Hong and Stein, 1999, p.2144) set up a list of "requirements" to justify theories departuring from full rationality and unlimited computational capacity of investors:

> "(1) [they have to...] rest on assumptions about investor behavior that are either a priori plausible or consistent with casual observation; (2) explain the existing evidence in a parsimonious and unified way; and (3) make a number of further predictions that can be subject to 'out-of sample' testing and that are ultimately validated."

Also it seems important to us that the departure takes only place in a very limited degree, that is, that as little new assumptions as possible are introduced, to make it easy to isolate the effect they produce. Therefore we try to stick to classical modelling tools where ever possible. In this part we already tried to convince you that our idea of improving heuristics satisfies criterium one. After setting up the model we will show that our results are consistent with major puzzling evidence observed in financial markets and thus satisfies point two, and later on we will make some "out-of-sample" predictions to address point three.

1.2 Literature

In our paper we focus on a maturing process of improving decision making over time, thus unexperienced traders should use a suboptimal heuristic for their decisions and improve it over time until they finally reach perfectly optimal behavior. To represent suboptimal decision making, our model takes a behavioral approach, and we are convinced that this point of view will allow for explanations of puzzles that can not be solved by purely classical theory in a satisfactory way. However there are of course different opinions about this and we do not want to deny that there are some good arguments for the other side also. One somewhat harsh critique of behavioral methods in economic theory can by found in (**?**). The cited paper surely hits some critical points. We nevertheless think that behavioral approaches will probably not be the magic bullet for all unsolved questions, but are worth to be as seriously considered as any classical attempt for the reasons listed above.

In the behavioral literature there are a couple of approaches to explain decision making that seems to be suboptimal from a classical point of view. Psychological research and experiments support most of these approaches in some way and it is not entirely clear until now which approach suites best for which problem set or whether a unifying theory could be formulated to incorporate all directions of this research area.

One approach of behavioral economics is to *alter the shape of utility functions* to incorporate psychological or behavioral findings in the process of decision making:
In 1979, Kahneman and Tversky (**?**) introduced their "prospect theory", an alternative form of utility function including the psychological fact that people seem to be "loss averse". The resulting function is not differentiable but has a kink at a certain reference point, making the subject excessive risk avers around this point. **?** (**?**) offered an related explanation of the equity premium puzzle first stated by **?** (**?**). They showed that by frequent portfolio evaluations the effect of the kink accumulates and makes an equity premium necessary to compensate for psychological costs due to the high risk aversion around the reference point. Another approach is the hyperbolic discounting applied by Loewenstein and Prelec (**?**), where preferences are inconsistent over time and actual decisions will be revised in the future. Work in that direction can also be found in (**?**) or in (**?**). Fehr and Schmidt (**?**) propose a utility function with an additional fairness term such that subjects bias their choice in a direction where payoff differences compared to a reference group are smaller. In Gebhardt XXX () a model is set up that allows diverging price paths in asset pricing by using Fehr-Schmidt preferences for investors.

Other approaches preserve the classical utility function but claim that *systematic "errors"* are made during the decision process. One approach of that kind is the mental accounting literature, e.g. Barberis and Huang (2001). Here it is argued that the same loss or gain can be valued differently depending on its framing. Another theory explains suboptimal behavior by investors overconfidence in private information. This idea was raised by Barber and Odean (**?**) and then used for example in a model by Daniel, Hirshleifer and Subrahmanyam (**?**) to explain over- and underpricing.

As a further example for systematic errors in a financial context, *momentum trading* is often cited. Momentum traders expect prices to be positively auto-correlated and thus buy when prices are rising. Of course in the long run this does not make much sense but actually there are empirical hints that prices are indeed short-run positively auto-correlated (see Cutler, Poterba and Summers, 1990) . Together with overconfidence (in identifying the right moment to leave the trend) it therefore seems imaginable that some traders might follow this strategy. A model where momentum traders are used along with "news watcher" is the one by Hong and Stein (Hong and Stein, 1999) . In their model, momentum traders are justified by the assumption that news spread slowly in the market and thus positive returns are possible by doing momentum trading. Hong, Lim and Stein (**?**) confirm this assumption by carrying out an empirical test to show that gradual information flow in markets exist. Jegadeesh and Titman (**?**) as well as Chan, Jegadeesh and Lakonishok (**?**)

propose some momentum strategies in their papers. Further discussions on this issue can also be found in (?).
From a psychological point of view, momentum trading can emerge from the tendencies of human observers to perceive patterns in a sample or time series where there actually are none. This effect ranges from simple optical delusion (e.g. seeing patterns in homogenous arrays of dots) to more complex fields as inferring trends from coincidental moves within a random walk, an issue widely discussed in (?). Griffin and Tversky (?) argue that this is due to humans tendency to overweight extremity to significance when observing real world data. They also formed the notion of *conservatism*, e.g. the inertia of belief on the arrival of new information. De Bondt (?) provides an empirical research based on 38,000 forecasts of stock prices and exchange rates and comes to the conclusion that *"non-experts"* expect the continuation of past "trends" in prices, whereas *"experts"* do not behave in such a way. Exactly this finding is the motivation for the choice in out model, that distinguishes unexperienced momentum- and experienced rational traders as well as different graduations in between.
Another strong indicator for real world momentum like trade is the existence of the large industry of technical analysis one can observe at the financial markets and which tries to forecast future price paths by past observation[1]. Clearly there must be some demand for this services to keep that industry alive.

In our model traders use a simple momentum strategy when first entering the market and then improve their strategy towards an "optimal" one over time. When classifying it, our model combines features of models with two types of traders as the Hong and Stein (1999) model mentioned above or the model by Grinblatt and Han (?) (interaction of a group of "rational" traders and a group with "disposition affected" traders) and models where investors change their view of the world over time as there is for example Barberis, Shleifer and Vishny (?), where representative investors consider the world switching between two "regimes", a mean reverting and a trend one, which follow a Marcov process. Also Hong and Stein (?) set up another model where investors switch between two forecasting models both of which use only a part of the available information. Investors change the model in use only if it did a particularly poor job in forecasting.

The new aspect in our model is that there are heterogenous groups of traders *and* they incrementally change their decision making strategies over time when getting experienced, such that each trader belongs to each group once in his lifetime. The improvement of decision strategies is only dependent of time, i.e. it is not triggered by observations or outside processes but purely by collecting experience in signal processing and it is incremental in a sense that it improves

[1] see Frankel and Froot (1990) on empirical evidence for the existence of so called "chartists".

from a pretty suboptimal momentum strategy to a fully rational fundamental signal observation strategy. Our motivation thereby is to fetch a gradual improve of investors behavior with respect to optimality.

2 The Model

To capture the ideas developed above we set up an multiperiod overlapping generations model with discrete time steps where agents improve the quality of their decisions over time. Each period, traders get signals from the fundamental process and they can observe past price changes. When first entering the market they are not experienced and behave in a suboptimal way by forming their beliefs on the basis of a momentum strategy. Next to possible rationalizations of momentum trading indicated in section **??**, the motivation behind this is that beginners are particularly susceptible to momentum behavior as they tend to follow the advise and the behavior of others since feeling unsure about their own judgement abilities.
When getting older and collecting experience, the traders learn to draw their own conclusions from the observed fundamental signal and older generations form beliefs more and more on a basis of signal observation, thus belief formation is heterogenous among generations. To capture the idea that traders regard direct signals as well as market observation in a momentum manner when forming their beliefs about the fundamental value of a risky asset, we consider the belief b_{it} as the probability assigned to high payoff by generation i at time t. It is composed out of two components b_{it}^s (signal component) and b_{it}^m (momentum component). The intermediate, non rational *"momentum belief"* (b^m) is motivated by the consideration "If I look at the history, I would judge the probability for high payoff to be...". The intermediate *"signal belief"* (b^s) belongs to the thought "If I interpret the signal I received, I would judge the probability of high payoff to be...". To compute the final belief b, both intermediate beliefs are regarded with weights depending on the age of traders. The explicit mechanism will be explained below.

2.1 Asset and Fundamental Process

There is a risky asset A on the market, which can be bought and sold by the agents once per period. The price is formed by the market clearing condition and the asset is in zero net supply, thus each unit purchased by an agent has to be sold short by someone else. The asset has a liquidating dividend payed out at the end of each period which can be high with probability π_t and low with probability $1 - \pi_t$. Every period t an exactly similar asset is set up with the volume of the liquidation return each time being $y_h > 0$ in the "good" case or $y_l \equiv 0$ in the "bad" case. The fundamental variable z_t is the log odds ratio of the high payoff:

$$z_t \equiv ln \frac{\pi_t}{1-\pi_t} \tag{1}$$

A $z_t > 0$ indicates $\pi_t > \frac{1}{2}$, $z_t < 0$ indicates $\pi_t < \frac{1}{2}$.

The fundamental value z can stay unchanged from $t-1$ to t with probability β_0 as well as increase or decrease by $+1$ or -1 with probabilities β_+ and β_- respectively, with $\beta_- = \beta_+ = \frac{1-\beta_0}{2}$. Whenever a change in z occurs, all agents receive the same informative common signal $s_t = \pm 1$ which is right (i.e. $s_t = \Delta z_t$) with probability $\tau > \frac{1}{2}$ or wrong ($s_t = -\Delta z_t$) with probability $1-\tau$.

π_t is calculated from the actual z_t by the inverse of (**??**)[2] :

$$\pi_t = \frac{1}{e^{-z_t}+1} \tag{2}$$

Next to investing in A, agents can invest in a riskfree asset B, yielding a payoff of r for each unit invested, where we assume $y_h > 2r > 0$, such that average payoff is larger for the risky asset than for the riskfree asset for $\pi = \frac{1}{2}$.

With no discounting the fair price of A for risk neutral investors would be

$$p_t^f = \pi_t y_h - r = \frac{1}{e^{-z_t}+1} y_h - r. \tag{3}$$

As agents are risk averse in our model the fair paying willingness p_t^* will be lower than p^f. It will be calculated below in section **??**.

2.2 Generations

The model is populated by n generations of agents. Each generation consists of a continuum of agents with mass 1 and should be thought of as a generation of traders in the market under consideration and not as a generation in a demographic sense. Thus, an "old" trader is one who entered the market long ago and had much time to collect experience in trading the related asset, whereas a "young" cannot refer to a trading history in that particulat market. In total there is a mass of n possible traders in the world. Each time step one generation enters the world and one is leaving it, therefore each generation lives for n time periods. As an endowment they start each period with the same W which is supposed to be some regular income from other activities. They can now form a portfolio, which is liquidated at the end of the period. The liquidation proceedings are not storable and are therefore consumed immediately.[3] For the sake of simplicity, agents do not discount future payoffs.

[2] Equation (**??**) and (**??**) are a way to transform a $z_t \in]-\infty; +\infty[$ to the interval $[0;1]$ of valid probabilities. It allows us to let z_t follow a random walk process without generating unfeasible values of π_t.

[3] By choosing this setting, we can avoid no consumption in the first period.

Until chapter **??**, where strategic trading is addressed, information about belief formation is private information, that is agents have no idea how other traders form their beliefs and assume they do it in the same way as themselves. Also, inter-period saving is not possible. However, at the beginning of the period there is no credit constraint to the agents and they can borrow to finance their portfolios, where debts have to be repayed during the consumption phase. Altogether as a result agents behave short-sighted up to chapter 4 and simply maximize the expected utility of their portfolios returns in a myopic way each period t. We restrict ourselves to this case at the moment to keep the model tractable as it is then possible to carry out analytical proofs. [4] In chapter **??** the restrictions are relaxed and agents can set their demands strategically and in a dynamic way, considering future effects of their choices.

All agents are risk-avers and have CARA utility functions of the style

$$U(x) = -e^{-\rho x} \tag{4}$$

with a constant coefficient of risk aversion $\rho > 0$ across generations.

2.3 Timing

Each period t consists out of three steps:

First, generations receive their endowment W and the fundamental signal s_t is revealed to all generations. The signal suggests whether the fundamental value z for high asset payoff has increased, decreased or stood unchanged. $s_t = 0$ denotes no change in the log odds ratio (this will be called *"no signal"* further on), $s_t = +1$ (if it wasn't a wrong signal) denotes an increase of z by one and $s_t = -1$ a decrease of z by one (*"positive"* and *"negative"* signal further on). If a signal arrives, it is used to update b^s in a bayesian way. A signal is only sent when a change occurs, thus no signal means there is no necessity to update b^s. Also, traders observe the recent price change Δp_t and interpret this as a signal of π to form their b^m.

Second, generations form beliefs about the expected payoff Ey_t and maximize $EU(x)$ accordingly, yielding a demand x_{it} of A for each period and each generation. The remainder $W_{it} - x_{it}$ is invested in the riskfree asset B. A market clearing price p_t is calculated.

Third, the actual z_t is revealed and a payoff y_t is drawn according to the probability π_t corresponding to z_t. All agents receive their liquidation returns and consume them. At the end of the turn all generations are getting older, the oldest generation drops out and a new generation of traders enters the market

[4] Moreover, in the actual setting where belief formation changes from period to period, dynamic optimization of an intertemporal consumption-savings problem would not make much sense for rationally restricted agents, as consistent beliefs about the own future actions would be nonsensical. In the strategic trading section, we will circumvent this problem by making a clearer distinction between rational and momentum traders.

with initial beliefs calculated from the observation of the recently revealed z_t and Δp_t.

2.4 Generations Belief Formation

2.4.1 Signal Belief b^s

As explained above, the direct signal observation belief b^s is deduced by updating the previous value π_{t-1} if a new signal arrives. π_{t-1} in turn can be exactly inferred by the observation of z_{t-1} at the end of period $t-1$.

The corresponding estimated z_t is: [5]

$$E^s(z_t) = z_{t-1} + s_t(2\pi_t - 1) \tag{5}$$

As the log odds ration function is not linear,

$$\begin{aligned} b_i^s = E^s(\pi_t) = \Big(\tau \frac{1}{e^{-(z_{t-1}+s_t)}+1} + (1-\tau)\frac{1}{e^{-(z_{t-1}-s_t)}+1}\Big) s_t^2 \\ +(1-s_t^2)\frac{1}{e^{-z_{t-1}}+1}. \end{aligned} \tag{6}$$

Equation **??** guarantees that b^s stays unchanged if no signal ($s_t = 0$) arrives, and is updated if $s_t \neq 0$.

2.4.2 Momentum Belief b^m

Similarly the momentum belief b^m is deducted by the observation of a possible recent price change as a signal on the log odds ratio of the asset. Therefore we define

$$\Delta p_t \equiv p_{t-1} - p_{t-2}. \tag{7}$$

Please note that Δp is defined such that Δp_t describes the increase in p from the *pre-preceding* to the *preceding* period and not from the preceding to the actual period! This is for the sake of conformance in time indexes in further equations.

In a typical momentum manner, estimates of fundamentals are linearly increasing in former price shifts. Therefore the point estimate for z_t, denoted by $\hat{z}_t$, is

$$\hat{z}_t = z_{t-1} + \gamma\Delta p_t. \tag{8}$$

Accordingly,

$$b_t^m = E_i^m(\pi_t) = \frac{1}{e^{-(z_{t-1}+\gamma\Delta p_t)}+1}, \tag{9}$$

[5] find the proof in the appendix

with the *momentum intensity parameter* γ describing the strength of the momentum effect, and $\gamma > 0$. For simplicity, we will set $\gamma = 1$ for the following sections. All results will be independent of γ, except the divergence/convergence behavior of cycles, and in section ?? we will study the impact of γ on dynamic optimization of rational agents exploiting the less experienced traders.

2.4.3 Belief Composition

The final belief b_i of generation i is composed out of b^s and b^m in dependence of the age of i. Younger generations are unexperienced in signal observation and rely to a higher degree on trend following strategies represented by the momentum belief b^m, with the youngest generation $i = 1$ being totally momentum driven and only regarding the momentum belief b^m. Older generations possess more "rational" capabilities and set their focus on b^s, with the oldest generation $i = n$ being perfectly rational and only regarding the signal belief b^s.

In general the final belief is formed as follows:

$$b_{it} = \frac{1}{n-1}\Big((n-i)b_t^m + (i-1)b_t^s\Big) \tag{10}$$

Each period, each generation is advancing to a more mature state, such that generation i becomes generation $i+1$. During this process, generations change their method of updating insofar as they now put a higher weight on signal observing and a lower weight on the observation of price changes. The oldest generation simply drops out and a new generation enters at position $i = 1$, whose initial beliefs are calculated by the observed z_{t-1} and Δp_t.

Keep in mind that this is a behavioral model and thus an agent is making the same systematic error (namely including the momentum term in his belief formation) again and again. His intern model of the world is unflexible in this sense and any deviations of the price from his predicted values are assigned to some noise and not to the quality of the own heuristic. A justification could be that people tend to repress wrong predictions they made. There is a large amount of psychological literature on this topic, e.g. (***citation). In the case of financial markets the trader might justify himself by seeing the asset as a long-term or growth investment, which - although it payed poor dividends today - will exhibit strongly increasing returns in future periods.[6] For a further discussion of this issue see for example (?, p.319f).
However, the new feature of our approach is, that the agents heuristic does get better over time. In the model presented here, this happens in a exogenous way. But one could well think of models where the size of estimation

[6]This would match the observation that growth markets - where the mentioned psychological effect would work best - seem to be more susceptible for bubble behavior than well established markets with lower expected growth.

errors determines the speed of the improvement of the belief formation process. Qualitatively the results should be the same in the later case, thus we stick to the mechanism described here to keep the model trackable.

2.5 Demand and Asset Price Formation

For the formation of asset prices we use a standard approach where agents can choose between a risky asset (the asset A under consideration) and a riskfree asset B. Agents estimate the liquidation value of the assets, where the certain payoff of one unit of B is r and the payoff of one unit of A is believed to be y_h with probability b_i and y_l with probability $1 - b_i$.

Each generation i maximizes the expected utility of its portfolio with respect to x_{it}, the amount of A bought at time t:

$$EU_{it}(x_{it}) = -b_{it}e^{-\rho(r(W-x_{it}p_t)+x_{it}y_h)} - (1-b_{it})e^{-\rho(r(W-x_{it}p_t))} \tag{11}$$

FOC:

$$\frac{\partial}{\partial x_{it}} = -b_{it}\rho(rp_t - y_h)e^{-\rho(r(W-x_{it}p_t)+x_{it}y_h)} - (1-b_{it})\rho(rp_t)e^{-\rho(r(W-x_{it}p_t))} = 0 \tag{12}$$

This yields

$$x_{it} = \frac{1}{\rho y_h} ln \frac{b_{it}(y_h - rp_t)}{(1-b_{it})rp_t} \tag{13}$$

Aggregating demand and equalling to the supply of 0 results in

$$\sum_{i=1}^{n} ln \frac{b_{it}(y_h - rp_t)}{(1-b_{it})(rp_t)} = 0 \tag{14}$$

$$ln \frac{(y_h - rp_t)^n \prod_{i=1}^{n} b_{it}}{(rp_t)^n \prod_{i=1}^{n}(1-b_{it})} = 0 \tag{15}$$

$$\frac{rp_t}{y_h - rp_t} = K_t \tag{16}$$

$$\Rightarrow p_t = \frac{K_t y_h}{r(1+K_t)} \tag{17}$$

with $K_t \equiv \sqrt[n]{\frac{\prod_i b_{it}}{\prod_i (1-b_{it})}}$.

Equation (**??**) can be seen as a general pricing function for two-asset models with CARA utilities and heterogenous beliefs. K can then be interpreted as

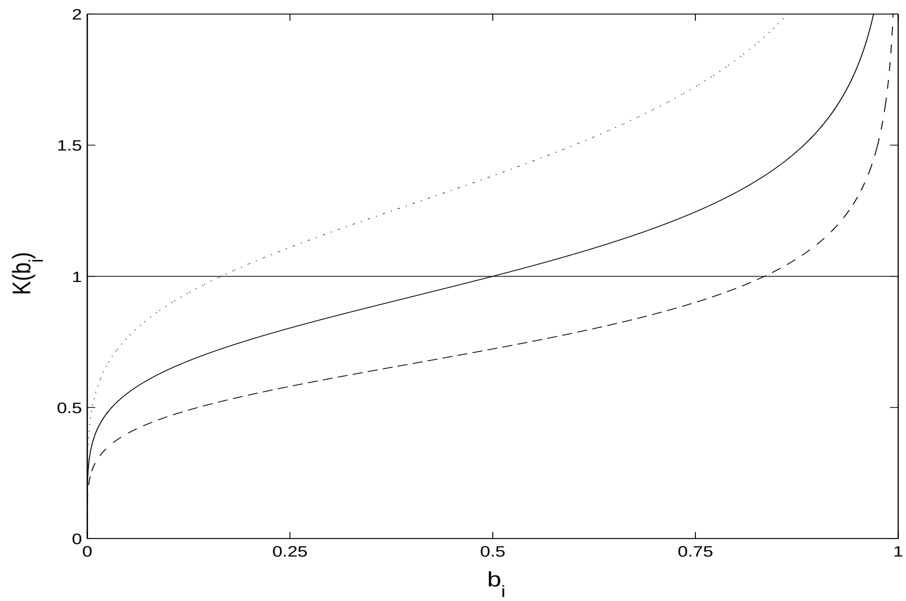

Figure 1: Sentiment function for n=5 dependent on belief b_i, with beliefs b_{-i} of all other agents $j \neq i$ keep fix at 0.5 (bold line), 0.4 (dashed line) and 0.6 (dotted line). Dependent on b_{-i}, i's belief has to be higher or lower in order to yield overall positive sentiment ($K > 1$).

the *market sentiment*. A $K > 1$ means overall belief in high payoffs is positive, thus $p > \frac{y}{2}$, $K < 1$ indicates negative overall belief implicating $p < \frac{y}{2}$. Very high K's represent strong overall belief in high future payoffs, leading to a price of y_h for $K \to \infty$, very low K's represent low aggregated beliefs in high future returns, thus yielding a price of 0 for A if $K \to 0$, as nobody wants to hold A if its expected value is 0.
Note also, that even if $K > 1$, still a majority of traders can assign probabilities of $b < \frac{1}{2}$ to high payoff if this is counterbalanced by extremely high beliefs of the minority. For example a single private belief of $0 + \epsilon$ or $1 - \epsilon$ for $\epsilon \to 0$ leads to a price of 0 resp. $\frac{y_h}{r}$ because the agent under consideration then is willing to sell short or buy the whole market if not counterbalanced by an appropriate extremely opposing belief of another agent.

We set up a *sentiment function* $K(b_i)$ which describes the market sentiment dependent on b_i if b_{-i}, i.e. all b_j with $j \neq i$, are held fix:

$$K(b_i) = \sqrt[n]{\frac{b_i \prod_{j \neq i} b_j}{(1 - b_i) \prod_{j \neq i}(1 - b_j)}} \tag{18}$$

For $n = 1$ or with homogenous beliefs and arbitrary n we get the pricing function for homogenous traders as a special case, thus it follows that $K_t = \frac{b_t}{1-b_t}$ and (**??**) simplifies to $p_t = \frac{b_t y_h}{r}$. Therefore, the fully rational price for correctly informed risk averse agents with a utility function described above would be

$$p_t^* \equiv \frac{\pi_t y_h}{r} = \frac{y_h}{r(e^{-z_t}+1)} \tag{19}$$

Thanks to the CARA-utility function, (??) and (??) are independent of W, that is, we face the same myopic decision problem each period, even if we would allow for accumulating wealth over periods.

2.5.1 Properties of the Demand-, Price- and Sentiment-Function

We now want to state some properties of the demand function (??), the price function (??) and the sentiment function (??). The price function (??) satisfies the requirements of being downward sloping in r and upward sloping in K and y, as

$$\frac{\partial p}{\partial r} = -\frac{K y_h}{r^2(1+K)} < 0, \tag{20}$$

$$\frac{\partial p}{\partial K} = \frac{y_h}{r(1+K)^2} > 0 \ \forall r > 0, K, y_h > 0 \tag{21}$$

$$\frac{\partial^2 p}{\partial K^2} = -\frac{2y_h}{r(1+K)^3} < 0 \ \forall r > 0, K, y_h > 0 \tag{22}$$

and

$$\frac{\partial p}{\partial y_h} = \frac{K}{r(1+K)} > 0 \ \forall r > 0, K > 0. \tag{23}$$

Individual demand x_i is downward sloping in p and upward sloping in b_i for prices smaller than $\frac{y_h}{r}$:

$$\frac{\partial x_i}{\partial p} = -\frac{1}{\rho p(y_h - rp)} < 0 \ \forall p < \frac{y_h}{r} \tag{24}$$

$$\frac{\partial x_i}{\partial b_i} = \frac{1}{\rho y_h b_i(1-b_i)} > 0 \ \forall 0 < b_i < 1, y_h > 0, \rho > 0 \tag{25}$$

Given the rational price of $p = \frac{\pi y_h}{r}$, individual demand would be positive if

$$\frac{b_i(y_h - rp)}{(1-b_i)rp} > 1 \tag{26}$$

$$\Leftrightarrow b_i > \pi \tag{27}$$

The slope of market sentiment K with respect to b_i is determined by:

$$\frac{\partial K}{\partial b_i} = \frac{K}{n b_i(1-b_i)}. \tag{28}$$

This is positive for all $b_i \in]0;1[$. Also:

$$\frac{\partial^2 K}{\partial b_i^2} = \frac{K(1-n+2nb_i)}{n^2 b_i^2 (1-b_i)^2} \tag{29}$$

$$\Rightarrow \frac{\partial^2 K}{\partial b_i^2} = 0 \Leftrightarrow b_i = \frac{n-1}{2n} \tag{30}$$

For smaller b_i, K is bent rightwards, for larger b_i leftwards in b_i. In any case, for $b_i > \frac{1}{2}$, $K(b_i)$ is leftward bend.

The total differential of K with respect to b is thus

$$dK = \sum_{i=1}^{n} \frac{\partial K}{\partial b_i} db_i \tag{31}$$

Suppose, that the probability pi_{t-1} for high payoff last period was $\frac{1}{2}$, i.e. we faced an asset with equal losing/gaining probability, and suppose further that the market was stable for at last two periods, i.e. no price shift was observed. Then, if a (correct/incorrect) fundamental signal arrives corresponding to a shift in π by $\Delta\pi$, generation i's belief is changed according to:

$$\Delta b_{it} = \pm \frac{i-1}{n-1} \Delta\pi(2\tau - 1) \tag{32}$$

Proof see appendix.

In this case and for small $\Delta\pi$, ΔK can be calculated from (**??**):

$$\Delta K \approx \sum_{i=1}^{n} \frac{\partial K}{\partial b_i} \Delta b_i = \sum_{i=1}^{n} \frac{iK\Delta\pi(2\tau-1)}{n^2 b_i (1-b_i)} \tag{33}$$

With initial $b_1 = b_2 = .. = b_n := b$ this simplifies to

$$\Delta K \approx \frac{(1+n)K\Delta\pi(2\tau-1)}{2nb(1-b)}. \tag{34}$$

The approximation is very accurate for small $\Delta\pi$ and becomes an equality for $\Delta\pi$ getting infinitesimal small.

2.6 Existence of bubbles

We now want to analyze the impact of a shock to π on the price path in our model.[7] We will show that a single positive shock to π and the according positive signal will lead to underpricing immediately after the observation of

[7] Think of a shock to the profitability of the asset, e.g. the arrival of a new production technology that increases expected payoffs.

the shock, to an overshooting price afterwards, a time of overpricing later on, and an endogenous breakdown of the asset pricing bubble and another period of underpricing. This cycle is repeated with diminishing amplitude until it converges to the new "fair" price defined by the altered π.
During the cycle, young generations loose money by buying when prices are high and old generations win money by selling short on high prices. The market will predominantly be held by young generations when prices fall and by old generations when prices rise. The overall lifetime performance of a trader depends mainly on the time he enters the market. If prices are low at that point in time, he is likely to obtain an overall gain from trading, if prices are high he probably will lose money.
Because the model displays a pretty complex dynamic as all former beliefs do matter for actual decisions via the momentum term, it is not possible to run proofs for price paths in *any* situation with arbitrary sequences of shocks. Nevertheless it is possible to show analytically how bubbles arise in formerly balanced markets and how prices fade into the new stable state after a number of cycles under certain conditions.
To set up a formal proof, we restrict ourselves to a situation where a formerly "balanced" market is hit by exactly one shock. The situation we have in mind is the arrival of a new technology, e.g. the information technologies in the late 90s, and its impact on the market. In section **??** we will model more complex price paths with several shocks by numerical simulation and we will show that bubbles will overlap and prices display excess volatility.

For the one-shock-case we assume a situation at time t, where all traders have neutral beliefs about π_t and beliefs are correct, i.e. $b_{it} = \pi_t = \frac{1}{2}$, $\forall i$. This implies a K_t of one and a price p_t equal to p_t^*. All generations hold zero units of the risky asset and it is assumed that p was constant for at least the last two periods, such there is no momentum in the market at time t.
First we will give the intuitions and informal proofs for under- and overpricing in period t respectively $t+1$ and we deduct limits for the change in K when a shock hits the formerly stable market in Lemma **??**. Later we will state formal proofs for under- and overpricing based on the estimations of ΔK and Δp in propositions **??** and **??**.

2.6.1 Intuition for Underpricing in Period t

In the situation described above, the risky asset will be underpriced compared to the fully rational price in the first period following a positive shock to the fundamental variable.
Consider a single shock Δz_t to the fundamental variable z and a corresponding change of π by $\Delta\pi$ (see equation **??**). Accordingly, traders receive a signal

s_t, which is assumed to be correct.[8] As p was constant the last periods, b^m obviously does not change at time t, but b^s increases and thus total b for generation i rises by Δb_i as defined in equation **??** due to the signal s_t.[9] For young generations the improve in b is pretty small (or even zero for generation $i = 1$), whereas it is high for old generations, with the oldest generation $i = n$ incorporating the full effect of the increase in b^s in their b_n. Clearly, the new price p_t will be above p_{t-1} (as almost all generations strictly increase their beliefs) and below p_t^*, as only the oldest generation increases its belief to the fully rational level. In other words, there will be underpricing in period t. Old generations will accumulate the risky asset as their demand will rise stronger while young generations will sell it short, assuming it is to expensive.

2.6.2 Intuition for Price Overshooting in Period $t + 1$

After p_t has formed in a way that clears the market, the true z_t is revealed to the traders. This has an effect on the signal beliefs b^s, as the uncertainty about z_t is dissolved, thus b_s rises to π_t. Also the basis for the estimation of b^m changes from z_{t-1} to z_t. But b^m is even further increased by the observation of the positive price change in the previous period. As a consequence, overall b will at least be equal to π_t and is strictly higher than π_t for all generations $i < n$. Therefore, generational demand for a price of p_t^* would be zero for generation n and positive for all generations $i < n$ (see equation **??**). As net supply is zero, this is impossible and price p_t must rise above p_t^*. In other words there will be overshooting prices in period $t + 1$.

As the maximal slope $\frac{1}{4}$ of $\pi(z)$ is reached for $z = 0$, the increase in b_{it+1} is bounded by:

$$\Delta b_{it+1} < \frac{n-i}{4(n-1)}(\Delta z_t + \Delta p_{t+1}) + \frac{i-1}{n-1}\Delta b^s_{t+1}. \tag{35}$$

It is particulary high for young generations, as they are heavily affected by the strongly positive momentum part. As their beliefs will now "overtake" the beliefs of old generations, they will accumulate the risky asset in this period whereas old generations will sell it at the new (and rationally too high) equilibrium price.

Further periods

In the consequent period $t + 2$, b^s does not change anymore, as no new signal arrives. The sole changes in beliefs will now stem from changes in b^m:

[8] if it was false, the first direction in the price path would be wrong. Nevertheless a bubble would arise later as the true value of z is revealed at the end of the period.

[9] find the proof in the appendix.

$$\Delta b_{it+2} < \frac{n-i}{4(n-1)} \Delta p_{t+2} \tag{36}$$

Note that, given b^s and z are constant, an increase in p at time $t+2$ only occurs if the change in prices was larger in the preceding period than in the pre-preceding period, thus if $\Delta p_{t+2} > \Delta p_{t+1}$. Therefore, to show that p will again increase in period $t+2$ it is sufficient to proof the increase in Δp in period $t+2$ relative to period $t+1$. We will do that below in Proposition **??**.

2.6.3 Formal Proofs for Under- and Overpricing

We will now set up a Lemma that defines the lower and upper bounds of changes in K in the first period after the arrival of a shock. It will be used for the successive proofs.

Lemma 1 *In a situation at time t, where all traders have neutral beliefs about π_t and beliefs are correct, i.e. $b_{it} = \pi_t = \frac{1}{2} \equiv b\ \forall i$ and p was constant for at least the last two periods, it holds that:*

$$2\Delta\pi(2\tau - 1) < \Delta K_t < \frac{\Delta\pi(2\tau-1)}{\frac{1}{2} - 2\Delta\pi^2(2\tau-1)^2} \tag{37}$$

Proof

From equation (**??**) and (**??**) and respecting the fact that $b_i = \pi\ \forall i$ at the beginning of the period and that $K(b_i)$ is convex for $b_i > \frac{1}{2}$ by (**??**), it follows that K_t will increase at least by

$$\Delta K_t > \sum_{i=1}^{n} \left(\frac{\partial K}{\partial b_i}\Big|_{b_{it}} \cdot \frac{i-1}{n-1} \Delta\pi(2\tau-1) \right) \tag{38}$$

$$= \sum_{i=1}^{n} \frac{(i-1)K\Delta\pi(2\tau-1)}{n(n-1)b_i(1-b_i)} \tag{39}$$

$$= \frac{K\Delta\pi(2\tau-1)}{2b(1-b)}. \tag{40}$$

With $K = 1$ and $b = \frac{1}{2}$ it follows:

$$\Delta K_t > 2\Delta\pi(2\tau-1). \tag{41}$$

Because $K(b)$ is convex, we also can deduct an upper bound for ΔK_t:

$$\Delta K_t < \sum_{i=1}^{n} \left(\frac{\partial K}{\partial b_i}\Big|_{b_{it}+\Delta b_{it}} \cdot \frac{i-1}{n-1} \Delta\pi(2\tau-1) \right) \tag{42}$$

$$= \sum_{i=1}^{n} \frac{(i-1)K\Delta\pi(2\tau-1)}{n(n-1)(b_{it}+\Delta b_{it})(1-b_{it}-\Delta b_{it})}, \tag{43}$$

where Δb_{it} denotes the increase in the beliefs of the particular generation.

Another (higher) upper bound can be derived in assuming that all signal beliefs are increased by the maximal amount Δb_n:

$$\Delta K_t < \sum_{i=1}^{n} \left(\frac{\partial K}{\partial b_i}\Big|_{b_{nt}} \cdot \frac{i-1}{n-1} \Delta\pi(2\tau - 1) \right) \tag{44}$$

$$= \sum_{i=1}^{n} \frac{(i-1)K\Delta\pi(2\tau-1)}{n(n-1)(b_{it} + \Delta b_{nt})(1 - b_{it} - \Delta b_{nt})}. \tag{45}$$

With $K = 1$, $b = \frac{1}{2}$ and (**??**) it follows that

$$\Delta K_t < \frac{\Delta\pi(2\tau - 1)}{2(\frac{1}{2} + \Delta\pi(2\tau-1))(\frac{1}{2} - \Delta\pi(2\tau - 1))} \tag{46}$$

$$= \frac{\Delta\pi(2\tau-1)}{\frac{1}{2} - 2\Delta\pi^2(2\tau-1)^2}. \tag{47}$$

□

The next lemma states the first aftershock-period price change triggered by the arrival of the shock.

Lemma 2 *Given the initial situation described in Lemma* **??***, p rises by less than* $\frac{y}{4r}\frac{\Delta\pi(2\tau-1)}{\frac{1}{2}-2\Delta\pi^2(2\tau-1)^2}$ *in period t.*

Proof

As $p(K)$ is concave, we can define the upper bound Δp_t in period t as follows:

$$\Delta p_t < \frac{\partial p}{\partial K}\Big|_{K_t=1} \cdot \Delta K_t. \tag{48}$$

From equations (**??**) and (**??**) it follows:

$$\Delta p_t = \frac{y}{4r} \frac{\Delta\pi(2\tau-1)}{\frac{1}{2} - 2\Delta\pi^2(2\tau-1)^2}. \tag{49}$$

□

Note also that the change of the rational price p^* in period t is

$$\Delta p_t^* = \frac{\Delta\pi y_h}{r}. \tag{50}$$

Now we set up propositions stating underpricing and overpricing following a shock.

Proposition 1 *Given the initial situation described in Lemma* **??**, *period* t *exhibits underpricing* at least *for all* $\Delta\pi$ *satisfying:*
$\Delta\pi < \frac{\sqrt{6-4\tau}}{4(2\tau-1)}$. [10]

Proof

Underpricing occurs if the increase in p is smaller than in p^*. Using equations (**??**) and (**??**) we get:

$$\frac{y}{4r}\frac{\Delta\pi(2\tau-1)}{0.5-2\Delta\pi^2(2\tau-1)^2} < \frac{\Delta\pi y_h}{r} \tag{51}$$

$$\Leftrightarrow \frac{(2\tau-1)}{2-8\Delta\pi^2(2\tau-1)^2} < 1 \tag{52}$$

$$\Leftrightarrow (2\tau-1) < 2-8\Delta\pi^2(2\tau-1)^2 \tag{53}$$

$$\Leftrightarrow \Delta\pi < \frac{\sqrt{6-4\tau}}{4(2\tau-1)}. \tag{54}$$

□

Despite describing only a subset of the solution, proposition **??** is already pretty general. If for example the signal is perfectly informative ($\tau = 1$), it follows the $\Delta\pi$ must be smaller than $\sqrt{\frac{1}{8}} \approx 0.35$ to satisfy proposition **??**. This would still be a fairly huge jump in the high payoff probability and it is yet only a lower bound. However this lower bound increases quickly when τ is falling only slightly, e.g. for $\tau = \frac{9}{10}$, $\Delta\pi$ must be smaller than 0.48 to fit Proposition **??**. This is already almost the maximal $\Delta\pi$ that is possible regarding that π starts at $\frac{1}{2}$. The massive increase in the boundary of valid $\Delta\pi$ is due to the fast rising slope of $\frac{\partial K}{\partial b}$ for high b's that results in high estimation errors in the estimation of Δp_t for high b's. Numerical simulations and the intuition displayed above show that in fact underpricing must occur for *all* possible $\Delta\pi$ and τ.

Similar to proposition (**??**) we derive a condition for $\Delta p_t < \frac{1}{2}\Delta p_t^*$:

Corollary 1 *Given the initial situation described in Lemma* **??**, *actual price* p_t *increases less than half as much as the rational price* p_t^* *in period* t, *if* $\Delta\pi < \frac{\sqrt{2-2\tau}}{2(2\tau-1)}$. *Stated differently:* $\Delta p_{t+1} < \frac{1}{2}(p_t^* - p_{t-1}^*)$.

Proof

Analog to proof of proposition **??** with an additional factor $\frac{1}{2}$ on the right hand side.

---

[10] However, this is only a lower bound and underpricing might well exist even for higher $\Delta\pi$. Actually, underpricing always takes place in period t, as described above.

□

Again, the result covers most of the possible parameter space. For $\tau = \frac{9}{10}$ it yields the condition $\Delta\pi < 0.279$, for $\tau = \frac{8}{10}$ a conditional $\Delta\pi < 0.52$. Thus, already for a τ as high as $\frac{8}{10}$, proposition **??** is fulfilled for *all* possible $\Delta\pi$. Once again this is a conservative estimate and the analytical proof that first period price increase will be less than half of rational price increase will hold for almost all cases except for very precise signals combined with high shifts in π.[11]

Proposition 2 *Given the initial situation described in Lemma* **??***, there will be overpricing in the risky asset in period $t+1$.*

PROOF

To give a formal proof of overpricing, note that it is necessary that

$$\begin{aligned} p_{t+1} &> p^*_{t+1} && (55)\\ \Leftrightarrow \frac{K_{t+1}y_h}{r(1+K_{t+1})} &> \frac{\pi_{t+1}y_h}{r} && (56)\\ \Leftrightarrow \frac{K_{t+1}}{1+K_{t+1}} &> \pi_{t+1} && (57)\\ \Leftrightarrow K_{t+1} &> \frac{\pi_{t+1}}{1-\pi_{t+1}}. && (58) \end{aligned}$$

Therefore it is sufficient to show that $b_{it+1} \geq \pi_{t+1}\ \forall i$ and $\exists i$ s.t. $b_{it+1} > \pi_{t+1}$:[12] As $b^s_{t+1} = \pi_t = \pi_{t+1}$ and $b^m_{t+1} > \pi_t\ \forall i < n$, this holds true.

□

After stating that there is underpricing in the first period following a positive shock, and price overshooting in the second, we now come to our main proposition, namely the description of the long run price path following a shock to fundamental variables.

Proposition 3 *Given the initial situation described in Lemma* **??***, at* least *for $\Delta\pi < \frac{\sqrt{2-2\tau}}{2(2\tau-1)}$ an asset pricing bubble will arise and deflate again after a single shock $\Delta\pi$ and an according signal with precision τ arrives at period t.*

[11] In numerical simulations it can be shown that corollary **??** in fact holds for the whole parameter range of τ and $\Delta\pi$.

[12] recall equation (**??**); this says that the odds ratio for all traders is at least as high as the rational odds ratio, with at least one trader having an higher odds ratio than the rational one.

PROOF

To proof the emergence of pricing bubbles, we will describe the development of the asset price p after a positive observed shock at the beginning of period t.

1st period after shock: Time t

As stated in corollary **??**, Δp_t will be smaller than $\frac{1}{2}\Delta p_t^*$, and underpricing will occur in the first period after the observed positive shock.

2nd period after shock: Time $t+1$

As shown in proposition **??**, period $t+1$ exhibits overpricing, thus $p_{t+1} > p_{t+1}^* = p_t^*$ and, combining Corollary **??** with underpricing in period t and overpricing in period $t+1$:

$$\Delta p_{t+2} = p_{t+1} - p_t > \Delta p_{t+1}. \tag{59}$$

In words the price increases more from period t to period $t+1$ than it increases from period $t-1$ (the initial price) to period t.

3rd period after shock: Time $t+2$

As the b^s stay constant from period $t+1$ to $t+2$ because all uncertainty about z was already abolished at the end of period t when z_t was revealed, and z did not change later, only b^m will matter for the overall belief formation of the generations.
b^m_{t+2} will increase compared to b^m_{it+1} if and only if Δp_{t+2} (which determines b^m_{t+2}) is larger than Δp_{t+1} (which determines b^m_{it+1}). Because this holds true by equation (**??**), b_i increase for all generations except generation $i = n$, thus K increases and subsequently p rises again and Δp_{t+3} is positive.

m^{th} period after shock: Time $t+m$

As the price function is asymptotically reaching $\frac{y_h}{r}$ for $K \to +\infty$ and $\Delta p_{t+1} > 0$, it is obvious that there exists a period m, such that: $\Delta p_m < \Delta p_{m-1}$. Thus in period m, p will decrease if no other signal arrives.[13] Another way to see this is to examine generational demand which increases slower and slower for high p even if beliefs are rising further (see equation **??**).

Further periods: Time $t+m+1, t+m+2, ...$

[13]Actually, simulations show, that in the whole parameter range it holds that $m = t+3$, that means already in $t+4$ prices are reverting again.

The decrease will be small in the first period and will trigger an accelerating downward cascade via the momentum term, as $\frac{\partial p}{\partial K}$ increases with falling K and also the slope of $\pi(\hat{z})$ is higher for $\hat{z}$ close to $\frac{1}{2}$. By the same argument as before, the decline in p will smoothen out when p is approaching the minimal price of zero, and another upward cascade will start. Depending on the value of γ, the cycle flattens out faster or slower, and p finally reaches the stable value p^*.

□

3 Numerical Simulation of Non-Strategic Price Paths

Although it was possible to proof that bubbles and a mispricing-cycle arise following a shock, it is hardly possible to state a closed form solution for an entire price path over more than a couple of periods due to the price dependency on all former prices and beliefs. On the other side it is easily possible to calculate price paths for given parameter settings. Thus, we ran a couple of numerical simulations to calculate explicit price path development after shocks.

3.1 The One Shock Case

As a first case we simulate a price path related to the situation described above in Lemma **??**, i.e. a single fundamental shock hitting the formerly balanced market. The parameter specification for this simulation is

number of generations $n = 5$
precision of signal $\tau = 0.7$
payoff of risk free asset $r = 1$
high payoff of risky asset $y_h = 3$
coefficient of risk aversion $\rho = \frac{1}{2}$
momentum intensity parameter $\gamma = 2$

3.1.1 Price Path

The simulation ran for 20 periods with the shock and the signal arriving in period 4. The result is displayed in figure **??**. Under- and overpricing as well as the cyclical price path can well be seen.

As stated in the intuition arguments in section **??**, young traders are loosing money on average, as they buy the market in its downturns, whereas old generations profit by selling their shares when prices are high and by buying when the price is falling.

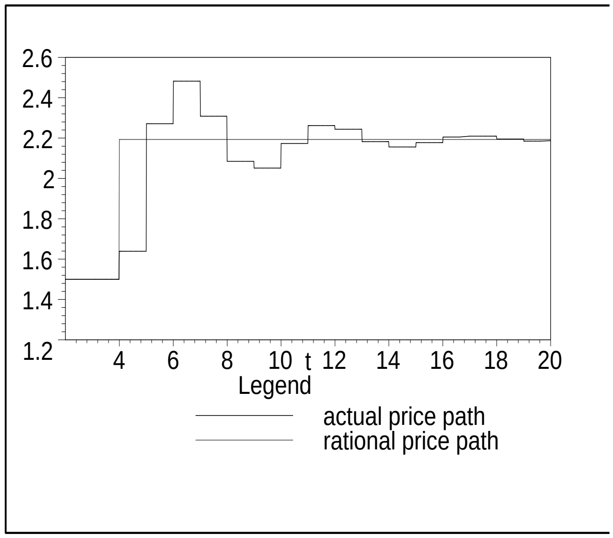

Figure 2: Simulated price path for one shock

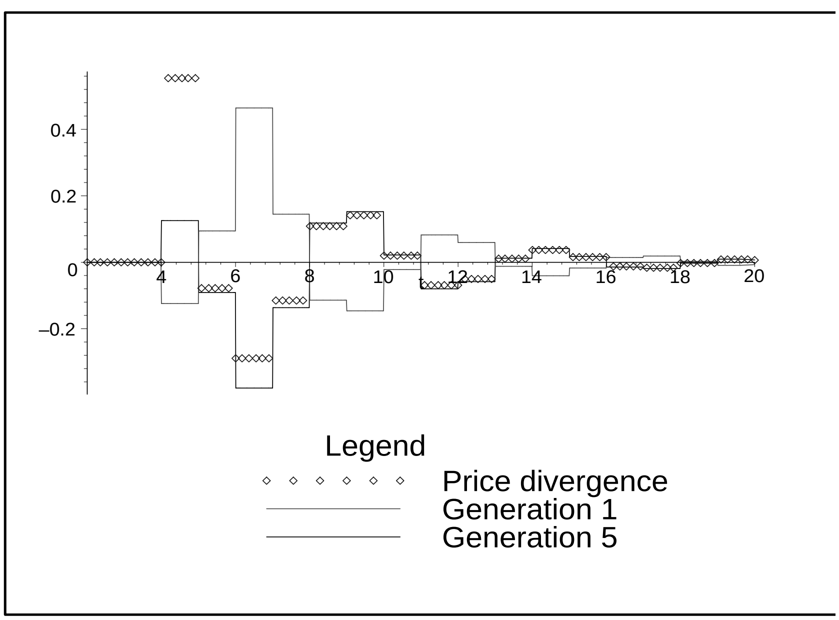

Figure 3: Generational demand and divergence of actual price from rational price

In our simulation we can track the amount of stocks held by a generations at any time. The result for the specifications given above is depicted in figure **??**, where the demand of generation 1 (youngest) and 5 (oldest) are shown. Next to that, the divergence of p_t from p_t^* is drawn, positive values indicate underpricing, negative values overpricing. As predicted by our considerations above, the old generation exhibits excess demand when prices are relatively low, whereas young generations demand the risky asset when its price is relatively high. To estimate the size of losses/gains made by young/old generations we track the per turn returns for each generation. For the liquidation return we set $E(\text{payoff, time t}) = \pi_t y_h$ in each period. From the numerical model we calculate the average gain/loss of each generation of traders and we find that indeed the young generations loose on average whereas the older win. However, the differences are only dramatic when there is substantial mispricing, i.e. in the first periods after the shock. In table 1, the average gains/losses per period are listed for all generations, considering ten periods after the arrival of the shock. Gains and losses are pretty small in this case, as there is only one shock. With many shocks in Section **??**, we will find the contrast increasing.

Table 1: Average per period loss/gain of generations

Generation #	loss/gain	in % of initial wealth
1	-0.0245	-2.45%
2	-0.0108	-1.08%
3	0.0011	+0.11%
4	0.0119	+1.19%
5	0.0223	+2.23%

In our model momentum traders lose money, and experienced investors profit from their existence. Yet they do not exploit their non rational behavior in a conscious way in the current reatment as we assumed that agents do not know anything about other investor's strategies, but only profit via price divergence from the rational path. Other models with heterogeneous investors often run into troubles justifying the survival of the non rational traders in the long run. One justification put forward by DeLong, Shleifer, Summers and Waldmann (**?**) is the idea that irrational noise traders generate extra risk for which they are compensated in return because short sighted and risk averse arbitrageurs cannot bet aggressively against them due to the risk that the former might push prices even further away from fundamental values next period.

In contrast, there is no need to justify survival of the "momentum traders" in our model, as they do not remain on the losing side for long, but gradually switch to the smart camp when getting older.

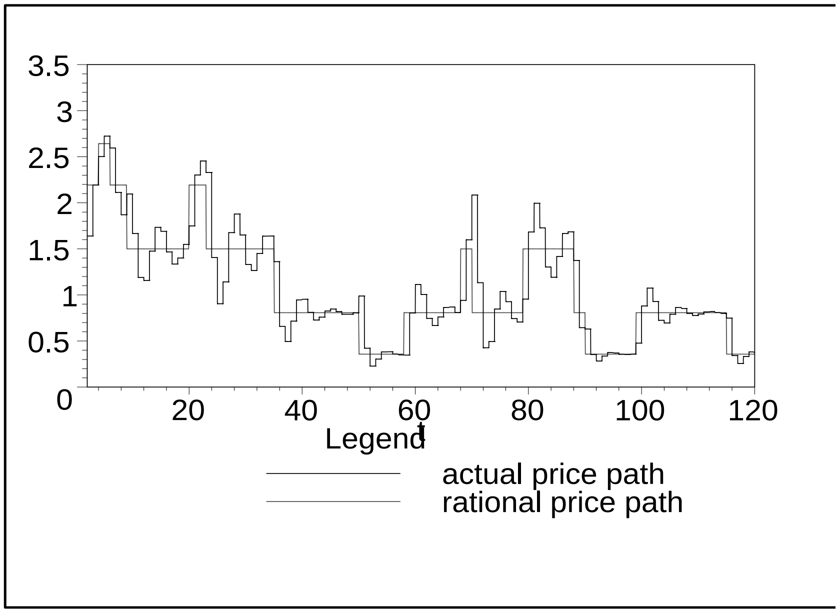

Figure 4: Simulated price path for multiple shock

3.2 The Multi Shock Case

In real asset markets, there are numerous shocks to the fundamental value of a stock over time. In our model, each shock triggers a bubble and a pricing cycle, and different cycles will overlap, if many shocks arrive. In figure **??** a situation is simulated where shocks arrive with a probability of 10% each period. All other parameters are as in section **??**. It is clearly visible that actual prices in principle follow the rational price, but that they show some excess volatility, a well known phenomenon in financial economics. Mispricing is especially severe for z_t around zero due to our process specification, where the changes in π_t are the higher, the closer z_t is to zero. One might criticize that the price path in our simulation is *following* the rational price path and thus the fundamental signals and not leading it. One possibly would expect the later, as financial markets are said to anticipate future developments. But note that we do not say anything about the nature of the fundamental shock. If you interpret it not as an direct shock but as a shock to expectations about fundamental variables, the simulated price path would indeed anticipate fundamental changes in future periods.

Analog to the one shock case, the corresponding demand functions for the oldest and the youngest generation are plotted for the first 40 periods in figure **??**.

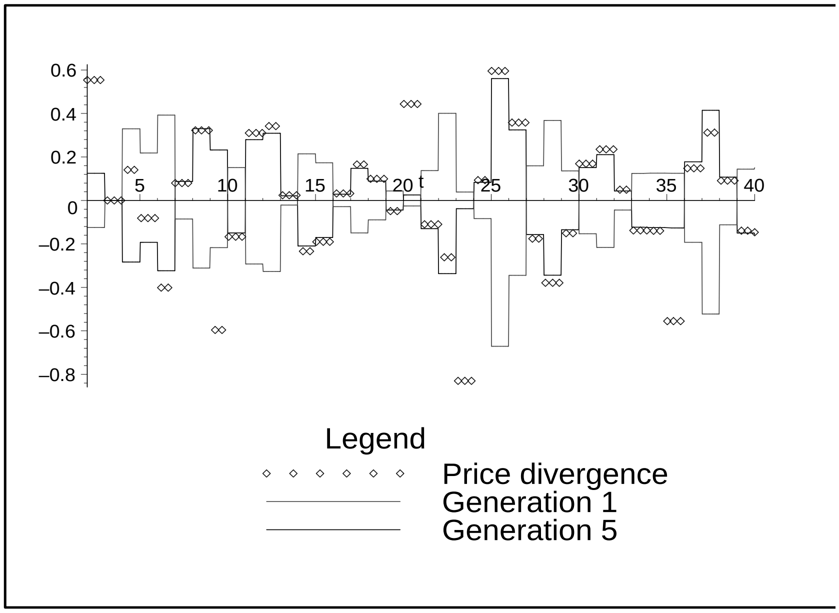

Figure 5: Generational demand and divergence of actual price from rational price for multiple shocks

4 Strategic Trading

In the following we will extend our model such that experienced traders do have information about the behavior of other traders. This is of course more realistic than our first case, but it will quickly lead to equations that are no more solvable in closed form. It will be possible to set up the maximization problem and to formulate first order conditions, but propositions about pricing behavior will only be possible by numerical computation respectively for specified parameter settings.
In the second part of this chapter we will introduce another change of the model settings. We will then assume that agents have only one initial endowment when entering the market and consume only at the end of their lifetime. We will refer to this setting as "last period consumption" in contrast to "consumption each period".

4.1 Consumption each period

Lets consider a market where the experienced, more "rational" traders actively exploit the bounded rationality of unexperienced traders. In contrast to the situation we analyzed before, they do not act as price takers anymore without knowing about other traders types, but they now are assumed to have full knowledge about the decision making process of the young traders. This seems to be a more natural assumption than before, as experienced traders have been in the "momentum camp" also when they were young and thus should be able to draw conclusions about young traders decision making processes. When a strategic trading setting makes experienced traders now finally entirely rational and is thus more realistic, it unfortunately is also much more complex and even less tractable than the situation with ignorant experienced traders. This is the reason we we restricted ourselves to the simpler setting until now.

We nevertheless want to show how introducing full rationality even amplifies our findings. Again we will have to pick out a simple case that on the one side makes clear the intuition behind this extension, and on the other does not lead into calculations too complex to solve in a meaningful way.
Our first restriction will be that we only allow for $n = 2$ generations and two groups of decision strategies to make a clear distinction between momentum ($\mathcal{M}$) and rational traders ($\mathcal{R}$). But we also want generations to stay more than 1 period in each group to allow for intertemporal optimizing of rational agents. Therefore we let each generation stay for two periods in each state and thus set the lifespan of each trader to $2n = 4$, i.e. each generation stays in the momentum group for 2 periods and then immediately switches to the rational group and stays there for another 2 periods.

4.1.1 Intuition

When optimizing in a dynamic way, rational agents now will not only put total weight on signal beliefs $b^s_{\mathcal{R}t}$ but also will try to exploit the systematical judgement errors made by unexperienced generations. The intuition for this is as follows:
Suppose we are in a balanced situation as defined in Lemma **??**. Now the positive shock arrives at time t and lets say the received signal s_t is right. Then prices will rise in period t due to the increase of $b^s_{\mathcal{R}t}$. Now there are two effect to be considered by the rational group $\mathcal{R}$ in the first period after the shock:

1) As $\mathcal{R}$ knows that $\mathcal{M}$ did not adjust their belief in response to the signal, they know that $\mathcal{M}$ will underestimate the value of A. Therefore $\mathcal{R}$ can strategically reduce its demand to lower p_t, as this decrease will not be fully absorbed by $\mathcal{M}$ and the gains from lower p_t will be higher than the losses from a smaller $x_{\mathcal{R}t}$ for a marginal decrease in $x_{\mathcal{R}t}$. This affects the immediate return at the end of period t. The optimal resulting price when $\mathcal{R}$ considers this effect only is denoted by $\hat{p_t}$ and will be called *"myopic exploiting optimum"* .[14]

2) As $\mathcal{R}$ knows that $\mathcal{M}$ will react to momentum in period $t+1$ by increasing its demand, there is an incentive to create as much momentum as possible in period t to make a maximum gain from selling stocks short next period.

[14]From now on we will take the standpoint of group $\mathcal{R}$, hence "optimal" means optimal from $\mathcal{R}$'s point of view.

Therefore, $\mathcal{R}$ will increase its demand s.t. p_t moves above the myopic exploiting optimum $\hat{p}_t$ to a point $\tilde{p}_t \geq \hat{p}_t$. It depends on the momentum intensity parameter γ whether there is a positive shift and whether even $\tilde{p}_t > p_t$, with p_t being the equilibrium price in the model without strategic trading. The price $\tilde{p}_t$ is called *"dynamic exploiting optimum"*.[15]

In the following we will set up the optimization equations.

4.1.2 Myopic Exploitation

As a benchmark case we first consider rational agents who do not care about future periods but optimize their utility myopically each period. As now only the first effect from above will work, we expect the price (compared to the non-exploiting case under same conditions) to be lower than without exploitation of unexperienced traders in periods where underpricing emerges and higher in periods where overpricing exists. To this benchmark we will later compare optimal prices when rational agents optimize in a dynamic way.

The rational agent $\mathcal{R}$ has to solve

$$\max_p EU_{\mathcal{R}}(p) \tag{60}$$

$$\text{s.t.}$$

$$x_{\mathcal{R}} = -x_{\mathcal{M}} \tag{61}$$

$$x_{\mathcal{M}} = \frac{1}{\rho y_h} ln \frac{b_{\mathcal{M}}(y_h - rp)}{(1 - b_{\mathcal{R}})rp} \tag{62}$$

The resulting maximization problem and the first order condition can be found in the appendix.
As it is not possible to solve the FOC for p analytically, we have to hark back to numerical methods to maximize (**??**). We use an algorithm based on golden section search and parabolic interpolation as described in (**?**). The used algorithm belongs to the group of constrained search algorithms and can be used because we know that prices outside the open interval $]0; \frac{y_h}{r}[$ can not be enforced by $\mathcal{R}$.[16] The algorithm then converges fast and with any given precision to the inner solution of the maximization problem.
Obviously the price will be independent of λ as $\mathcal{R}$ ignores future periods at this stage of the model. In figure **??** the price is depicted for $\tau = 0.7$ and as

[15] When considering strategic behavior, we treat $\mathcal{R}$ as on actor in this chapter, therefore assuming that rational traders are able to cooperate perfectly. If they do not, it depends on the actual size and number of agents in this group whether the price impact is large or small. At least it seems plausible, that - even if agents do not collude - in some markets there is strategic power of single actors. As an example one could think of firm stock where single holders like other firms own considerable amounts of the available shares.

[16] for $p \to 0$ the demand of $\mathcal{M}$ rises to infinity, for $p \to \frac{y_h}{r}$ to minus infinity.

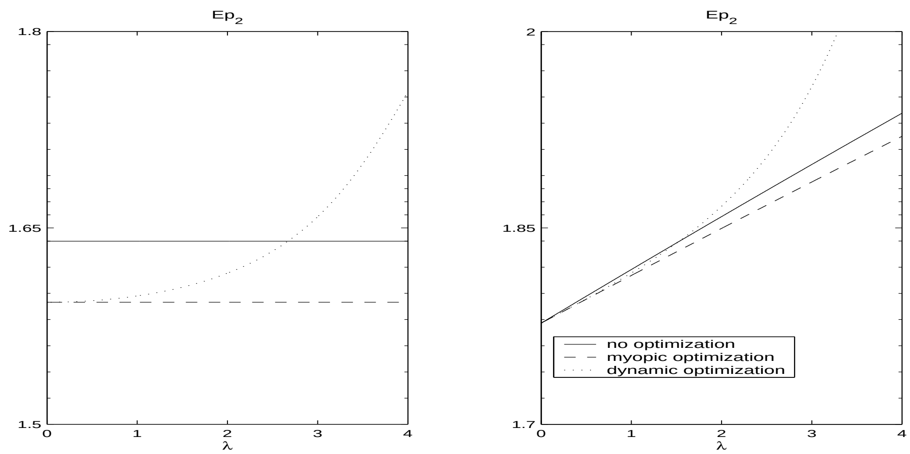

Figure 6: Prices p_1 and p_2 for different λ with no optimization, myopic and dynamic optimization by rational agents $\mathcal{R}$.

expected, it is below the non-exploiting price p_n.
In the second period the optimal myopic price will be higher than without exploiting rational agents because now the intuition from above will work the other way round [17]. Depending on λ the optimal price will increase in τ. A plot of second period myopic optimal pricing can also be found in figure **??**.
In figure **??** the corresponding expected utilities are depicted for different λ. Note that optimizing in a myopic way yields positive gains in period 1, but also a loss in period 2, as the momentum effect is weakened when prices are pushed down in the first period. Thus, for $\lambda > \hat{\lambda}$, total utility is actually higher with no optimization than with myopic optimization.
In the appendix, the solutions for different τ are plotted and it can be seen that the optimal price p_1 increases linearly in τ.

4.1.3 Dynamic Optimization, Consumption Each Period

If $\mathcal{R}$ considers the consequences of his action in period 1 on prices in period 2 he additionally has to consider the second effect stated above. We then expect prices to be strictly above the myopic maximizing prices for $\lambda > 0$, and exactly matching the latter for $\lambda = 0$.

Therefore, in period 1, a dynamically optimizing rational trader who has one more period to life faces the optimization task

[17] note than now we have overpricing.

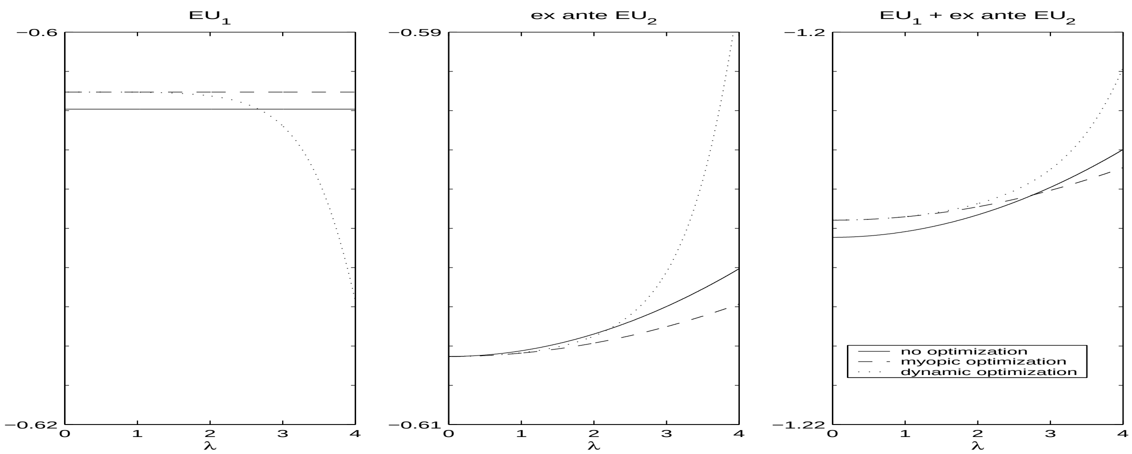

Figure 7: Period 1, period 2 and total expected utilities depending on λ for no optimization, myopic and dynamic optimization by rational group $\mathcal{R}$.

$$\max_{p_t} \sum_{k=t}^{t+1} EU_{\mathcal{R}}(x_{\mathcal{R}k}, p_k, p_{k-1}, p_{k-2}) \tag{63}$$

s.t.

$$x_{\mathcal{R}k} = -x_{\mathcal{M}k} = \frac{1}{\rho y_h} ln \frac{b_{\mathcal{M}k}(y_h - rp_k)}{(1 - b_{\mathcal{R}k})rp_k} \tag{64}$$

$$b_{\mathcal{M}t+1} = \frac{1}{e^{-z_t - \gamma(p_t - p_{t-1})} + 1}. \tag{65}$$

Thus, given p_{t-1} he faces a two dimensional maximization problem where he has to choose an optimal triple $(p_t, \overline{p}_{t+1}, \underline{p}_{t+1})$, where $\overline{p}_{t+1}$ respectively $\underline{p}_{t+1}$ are the optimal myopic exploiting prices in period $t+1$ for the case that at the end of t the signal s_t turns out to have been correct or incorrect. [18]

Stated differently, in period one (set $t \equiv 1$) he has to choose $x_{\mathcal{R}1}$ such that his demand induces a price $\tilde{p}_1$ that satisfies the equation:

$$\frac{\partial EU_{\mathcal{R}1}(\tilde{p}_1)}{\partial p_1} + \frac{\partial EU_{\mathcal{R}2}(\tilde{p}_1)}{\partial p_1} = 0. \tag{66}$$

The second period marginal ex ante expected utility for an change in p_1 can be found in the appendix. For the first period marginal expected utility with respect to p_1, the derivative of (**??**) as stated in equation **??** in the appendix can be used, as there is no momentum effect in the first period.

[18] Note that the rational exploiting agent would optimize differently in period $t+1$ dependent of the actual realization of z_t and that he chooses myopic optimization in $t+1$, as this is his last period to live.

The resulting equation is somewhat unhandy and probably cannot be solves in a closed form. But again, it is pretty easy to calculate the marginal) expected utilities for given parameters and then find the solution of (**??**) using a numerical algorithm, as it turns out (and is clear by intuition) that marginal expected utility of period 1 is downward sloping for prices $p_1 > \hat{p}_1$, coming from a value of zero for $p = \hat{p}_1$ and marginal expected utility of period 2 is downward sloping for $p_1 > \hat{p}_1$, starting from a strictly positive value for $p = \hat{p}_1$ and $\lambda > 0$.[19] Thus, a simple incremental increase in p_1 and a refining of steps close to the solutions leads to a fast and accurate convergence to $\tilde{p}_1$. Optimized prices can be found in figure **??**, corresponding expected utilities in figure **??**. Price p_1 will be chosen equal to myopic optimization for no momentum and will be strictly higher for a positive momentum effect ($\lambda > 0$). But in contrast to λ vs. p_2, the relationship of λ and p_1 is not a monotonic one, as for very high λ additional momentum will yield only small additional returns because p_2 will then be close to the maximum anyway such that costs in period 1 will outweight gains in period 2. This can not be seen in figure **??** but in figure XXX in the appendix, where the curves are plotted for a wider range of λ. Expected utility goes down in period 1 for increasing λ as prices are chosen higher. For the same reasons as above, the relationship is non-monotonic. Expected utility of period 2 rises monotonically in λ as well as total expected utility.

...work in progress...

[19]for $\lambda = 0$ marginal expected utility of period 2 is zero.

A Proof of Equation (??) - $E(z_t)$

Let z^+ and z^- denote $z_{t-1} + s_t$ and $z_{t-1} - s_t$, i.e. z^+ is the value of z at time t if the signal was right, z^- is the value of z_t if the signal was wrong. The probability for s_t being right is denoted by τ. According to Bayes rule,

$$pr(z^+|s_t) = \frac{pr(z^+|s_t)}{pr(z^+|s_t) + pr(z^-|s_t)} = \frac{\tau}{\tau + (1-\tau)} = \tau$$

and

$$pr(z^-|s_t) = \frac{pr(z^-|s_t)}{pr(z^-|s_t) + pr(z^+|s_t)} = \frac{1-\tau}{(1-\tau) + \tau} = 1 - \tau.$$

Therefore:

$$E(z_t) = \tau z^+ + (1-\tau)z^- = \tau(z_{t-1} + s_t) + (1-\tau)(z_{t-1} - s_t) = z_{t-1} + s_t(2\tau - 1).$$

□

B Proof of Equation (??) - Δb_{it}

Let π^+ and π^- denote the new π-values for the case that the received signal s_t was correct and z changed from $z_{t-1} = 0$ to $z_t = s_t$. $\Delta\pi$ be the absolute change in π which is equal for an increase and a decrease in z if z_{t-1} was zero. Note that $b_{it-1} = \pi_{t-1}$.

Generation i's belief is then

$$b_{it} = \frac{i-1}{n-1}(\tau\pi^+ + (1-\tau)\pi^-) + \frac{n-i}{n-1}\pi_{t-1} \quad \text{(B.67)}$$

$$= \pi_{t-1} + \frac{i-1}{n-1}\Delta\pi(2\tau - 1). \quad \text{(B.68)}$$

Hence,

$$\Delta b_{it} = b_{it} - \pi_{t-1} = \frac{i-1}{n-1}\Delta\pi(2\tau - 1). \quad \text{(B.69)}$$

□

C Optimization problem of myopic exploiting rational agents

The optimization problem is thus:

$$\max_p -b_{\mathcal{R}} e^{-\rho(rW - \frac{1}{\rho y_h} ln(\frac{y_h}{rp} - 1)(y_h - p))} - (1 - b_{\mathcal{R}}) e^{-\rho(rW + \frac{1}{\rho y_h} ln(\frac{y_h}{rp} - 1)p)}. \quad \text{(C.70)}$$

The marginal expected utility in period one after stable prices for two periods ($p_0 = p_{-1}$), dependent on p_1 is:

$$\frac{\partial EU_1(p_1)}{\partial p_1} = b_{\mathcal{R}1}\Big(\frac{y_h - p_1}{rp_1^2(\frac{y_h}{rp_1}-1)} + \frac{1}{y_h}ln(\frac{y_h}{rp_1}-1)\Big)e^{-\rho rW - \frac{1}{y_h}ln(\frac{y_h}{rp_1}-1)(y_h-p_1)} + (1-b_{\mathcal{R}1})\Big(-\frac{1}{rp_1(\frac{y_h}{rp_1}-1)} + \frac{1}{y_h}ln(\frac{y_h}{rp_1}-1)\Big)e^{-\rho rW + \frac{1}{y_h}ln(\frac{y_h}{rp_1}-1)p_1}, \quad \text{(C.71)}$$

where

$$b_{\mathcal{R}1} = \tau\frac{1}{e^{-s_1}+1} + (1-\tau)\frac{1}{e^{s_1}+1}.$$

D Optimization problem of dynamic exploiting rational agents

From an *ex ante* perspective (uncertainty about first period z_1), the second period expected utility dependent on p_1 is:

$$EU_2(p_1) = \tau(\overline{b}_{\mathcal{R}2}U(s_1,y_h) + (1-\overline{b}_{\mathcal{R}2})U(s_1,0)) + (1-\tau)(\underline{b}_{\mathcal{R}2}U(-s_1,y_h) + (1-\underline{b}_{\mathcal{R}2})U(-s_t,0)), \quad \text{(D.72)}$$

where $U(c,d)$ is the myopic maximized utility when $z_1 = c$ and $y_2 = y_h$. For example $U(-s_1,y_h)$ is the maximal possible utility of generation $\mathcal{R}$ if it exploits $\mathcal{M}$, the signal was wrong ($z_1 = -s_1$) and the risky asset A yields high payoff at the end of period 2 ($y_2 = y_h$).

Therefore,

$$\frac{\partial EU_2(p_1)}{\partial p_1} = \frac{r\tau\overline{p}_2(1-\overline{b}_{\mathcal{M}2}(p_1))}{y_h\overline{b}_{\mathcal{M}2}(p_1)(y_h - r\overline{p}_2)} \cdot \Big(\frac{\overline{b}'_{\mathcal{M}2}(p_1)(y_h - r\overline{p}_2)}{(1-\overline{b}_{\mathcal{M}2}(p_1))r\overline{p}_2} + \frac{\overline{b}_{\mathcal{M}2}(p_1)(y_h - r\overline{p}_2)\overline{b}'_{\mathcal{M}2}(p_1)}{(1-\overline{b}_{\mathcal{M}2}(p_1))^2 r\overline{p}_2}\Big) \cdot \Big[-\overline{b}_{\mathcal{R}2}(y_h - \overline{p}_2)e^{-\rho rW - \frac{1}{y_h}ln\frac{\overline{b}_{\mathcal{M}2}(p_1)(y_h - r\overline{p}_2)}{(1-\overline{b}_{\mathcal{M}2}(p_1))r\overline{p}_2}(y_h-\overline{p}_2)} + (1-\overline{b}_{\mathcal{R}2})\overline{p}_2 e^{-\rho rW + \frac{1}{y_h}ln\frac{\overline{b}_{\mathcal{M}2}(p_1)(y_h - r\overline{p}_2)}{(1-\overline{b}_{\mathcal{M}2}(p_1))r\overline{p}_2}\overline{p}_2}\Big] - \frac{(1-\tau)r\underline{p}_2(1-\underline{b}_{\mathcal{M}2}(p_1))}{y_h\underline{b}_{\mathcal{M}2}(p_1)(y_h - r\underline{p}_2)} \cdot \Big(\frac{\underline{b}'_{\mathcal{M}2}(p_1)(y_h - r\underline{p}_2)}{(1-\underline{b}_{\mathcal{M}2}(p_1))r\underline{p}_2} + \frac{\underline{b}_{\mathcal{M}2}(p_1)(y_h - r\underline{p}_2)\underline{b}'_{\mathcal{M}2}(p_1)}{(1-\underline{b}_{\mathcal{M}2}(p_1))^2 r\underline{p}_2}\Big) \cdot \Big[\underline{b}_{\mathcal{R}2}(y_h - \underline{p}_2)e^{-\rho rW - \frac{1}{y_h}ln\frac{\underline{b}_{\mathcal{M}2}(p_1)(y_h - r\underline{p}_2)}{(1-\underline{b}_{\mathcal{M}2}(p_1))r\underline{p}_2}(y_h-\underline{p}_2)} + (1-\underline{b}_{\mathcal{R}2})\underline{p}_2 e^{-\rho rW + \frac{1}{y_h}ln\frac{\underline{b}_{\mathcal{M}2}(p_1)(y_h - r\underline{p}_2)}{(1-\underline{b}_{\mathcal{M}2}(p_1))r\underline{p}_2}\underline{p}_2}\Big], \quad \text{(D.73)}$$

where

$$
\begin{aligned}
\overline{b}_{\mathcal{M}2}(p_1) &\equiv \frac{1}{e^{-s_1-\lambda(p_1-p_0)}+1} \quad (b_{\mathcal{M}2} \text{ dependent on } p_1 \text{ if } s_1 \text{ was right}),\\
\underline{b}_{\mathcal{M}2}(p_1) &\equiv \frac{1}{e^{+s_1-\lambda(p_1-p_0)}+1} \quad (b_{\mathcal{M}2} \text{ dependent on } p_1 \text{ if } s_1 \text{ was wrong}),\\
\overline{b}'_{\mathcal{M}2}(p_1) &= \frac{\lambda e^{-s_1-\lambda(p_1-p_0)}}{(e^{-s_1-\lambda(p_1-p_0)}+1)^2},\\
\underline{b}'_{\mathcal{M}2}(p_1) &= \frac{\lambda e^{+s_1-\lambda(p_1-p_0)}}{(e^{+s_1-\lambda(p_1-p_0)}+1)^2},\\
\overline{p}_2 &\equiv argmax_{p_2} EU_2(p_2|p_1, s_1 = +\Delta z_1) \quad (\text{optimal } p_2 \text{ if } s_1 \text{ was right}),\\
\underline{p}_2 &\equiv argmax_{p_2} EU_2(p_2|p_1, s_1 = -\Delta z_1) \quad (\text{optimal } p_2 \text{ if } s_1 \text{ was wrong}),\\
\overline{b}_{\mathcal{R}2} &\equiv \frac{1}{e^(- s_1) + 1} \quad (b_{\mathcal{R}2} \text{ if } s_1 \text{ was right}),\\
\underline{b}_{\mathcal{R}2} &\equiv \frac{1}{e^(+ s_1) + 1} \quad (b_{\mathcal{R}2} \text{ if } s_1 \text{ was wrong}).
\end{aligned}
$$

In the square brackets you can find the four possible end state utilities again.

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