Private Provision of a Complementary Public Good*

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Preliminary and Incomplete Version

Abstract

For several years, an increasing number of firms have begun to invest in open source software (OSS). While improvements in the OSS are not appropriable, companies can benefit indirectly in a complementary proprietary segment. In this paper we study this incentive for investment in OSS. In particular we ask how market entry affects the incentive to contribute to the public good. Surprisingly, we find that there exist cases where the incumbent firms may benefit from market entry. Moreover, we can show that the equilibrium with an increasing numbers of firms does not converge to the price-taker equilibrium.

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1 Introduction

For several years, an increasing number of firms like IBM and Hewlett-Packard or Suse and Red Hat have begun to invest in Open Source Software. Open Source Software, such as Linux, is typically under the General Public License (GPL) and therefore any improvement must be provided for free. Hence, an Open Source Software can be seen as a non-excludable public good meaning that firms are not able to sell the Open Source Software or their improvements. This issue raises the question why companies do contribute to such a public good.
Lerner and Tirole (2000) argue that the firms expect to benefit from their expertise in some market segment of which the demand is boosted by the introduction of a complementary open source program. Although the companies cannot capture directly the value of an open source program's improvement they can profit indirectly through selling more complementary proprietary goods at a potentially higher price.

In this paper we study this incentive of investment in Open Source Software. In particular we ask how market entry and therefore tougher competition affects the incentive to contribute to the public good. In their paper Lerner and Tirole (2000) address this issue:

Because firms do not capture all the benefits of the investments, the free-rider problem often discussed in the economics of innovation should apply here as well. Subsidies by commercial companies for open source projects should remain limited unless the potential beneficiaries succeed in organizing a consortium (which will limit the free-riding problem).

Does this mean that a monopolist always provides a larger amount of the public good than a company which faces competition? What is to say about the reaction to tougher competition, considering the total production of the public and private good, the profits and the social surplus?

We contribute to answering these questions by analyzing a model with Cournot-Competition where the firms can produce a private and a public good, but can sell only the private good. For the consumers, the private and public goods are complements. An increase in the total amount of the public good increases their willingness to pay for the private good.

We show that the market entry of an additional firm has a positive externality (if the entrant contributes to the public good) and a negative externality (through the entrant's production of the private good) on the incumbents. We find that for certain cost and demand functions each firm reduces its output as a consequence of market entry and suffers a decrease in profits. In this case incumbents dislike market entry. Surprisingly, for certain cost and demand constellations, it is also possible that with a market entry every firm expands its output and is able to increase its profit. However, a social planer unambiguously prefers market entry. Finally, we show that with an increasing number of firms the results do not converge to a market equilibrium with perfect competition in which the firms act as price taker and where therefore nobody should invest in the public good.

This article is related to the public good literature that is concerned with the private provision of a non-excludable public good. In standard models of public good provision, households can buy the private and the public good at constant prices. Therefore, this approach is equivalent to a setup where the individuals can produce the private and the public good at constant costs. Their decision to buy one unit of the public good is equivalent to the decision to produce one unit of the public good whereby they receive directly a certain utility from the production of the public good.
In our model, firms do not benefit directly form the production of a public good. They produce the public good because of the complementarity to the private good.
This leads to a new effect: The incentive of the firms to contribute to the public good depends on the market environment and therefore on the number of competitors. In the normal setup the marginal utility of the public good is

determined through the utility function and exogenously given. In our setup it depends on the ability to use the additional unit of the public good to earn money in the proprietary sector. And this depends on the severity of competition in this sector.

A second strand of literature our paper is related to is the literature of Multimarket Oligopoly. Bulow, Geanakoplos and Klemperer (1985) analyze the effects of a change in one market environment. In their model different markets are related because due to the production technology the costs are interrelated across these markets. In our model the market are not related through the production technology but through the demand function. Bulow, Geanakoplos and Klemperer (1985) address this issue but do not formalize it. They see that firms must consider cross-effects in making marginal revenue calculations and consider the strategic effects of their actions in one market on competitors' actions in a second. In our model we formalize this issue and isolate the different effects, whereby we have the additional interesting feature that there exists no price in the public good market.
Becker and Murphy (1993) analyze a model in which advertisement and an advertised good enters the utility function of the households. Advertisement has the property that it raises the willingness to pay for the advertised good and is from the economic viewpoint complementary to the advertised good. Nevertheless, in their setup a firm's advertisement is only complementary to its own private good. In our model it is also complementary to the competitors' private goods.

We will proceed as follows. The next section sets up the general model. In Section 3 we use a linear demand function and quadratic cost functions to illustrate the effects of the market entry on different variables like e.g. profit, prices and social surplus. In Section 4 we analyze the general adjustment process of a firm's production in response to an exogenous change in the amount produced by other companies. The final section concludes.

2 The Model

Consumer

Consider individuals who consume two goods, x and y. Let x describe the private good, e.g. hardware or service, and y the non-excludable public good, e.g. Open Source Software. These two types of goods are complements for the consumers. The consumers have to pay a price p for every unit of the private good x they want to consume. For the non-excludable public good y there is by definition no price to pay. Therefore every individual consumes the whole amount available of the public good.

The total amount of the private good produced and consumed is described by X and the total amount of the public good by Y. In order to analyze the described problem we introduce the following inverse demand function:

$$p = p(X; Y) \tag{1}$$

with

$$\frac{\partial p}{\partial X} < 0 \tag{2}$$

$$\frac{\partial p}{\partial Y} > 0 \tag{3}$$

The inverse demand function is as usual decreasing in x because the marginal utility of the private good is declining. Since the public and private goods are complements the consumers' willingness to pay for the private good is increasing in y. For example, the performance of a server depends crucially on the ability of the server operating system to use the power of the hardware. If the quality of the operating system is increasing without generating any costs for the consumers, their willingness to pay for the server increases due to the better performance.

Firms

Firm i ($i \in \{1, 2, ..., N\}$) can produce x_i and y_i at costs of $K_i(x_i)$ and $K_i(y_i)$. Let X_{-i} and Y_{-i} describe the produced amount by all other firms. Therefore

$$x_i + X_{-i} = X \text{ and } y_i + Y_{-i} = Y$$

We assume that the firms are engaged in a Cournot-Competition. All firms decide simultaneously how much of the private good and of the public good they produce. Hence, the profit function for firm i is

$$\pi_i = x_i * p\left(x_i, X_{-i}, y_i, Y_{-i}\right) - K_i(x_i) - K_i(y_i) \tag{4}$$

3 The Analysis of Market Entry with Linear Demand and Quadratic Cost Functions

In this section we restrict our attention to the case of an inverse demand function that is linear in x and y.

$$p = A - X + Y \tag{5}$$

The implications of a more general demand function are studied in the section 4.

Through this simplification we are able to derive closed-form solutions that allow us to address the following questions: How do the profits and the production of the incumbent firms change? What are the effects on welfare and on prices?

Consider N symmetric firms and assume for all firms quadratic cost functions.

$$K(x) = dx^2 \tag{6}$$

$$K(y) = fy^2 \tag{7}$$

The parameters d and f represent the weight of the cost functions with which they influence the firm's profit.

We assume that in an equilibrium all competitors produce the same amount of the private good x and the same amount of the public good y. Thus profits for firm i, whereby x_j and y_j represents the amount produced by firm $j \in \{1, ..., N\}$ with $j \neq i$, becomes

$$\pi_i = x_i * (A - x_i - (N-1)x_j + y_i + (N-1)y_j) - dx_i^2 - fy_i^2 \tag{8}$$

The first-order conditions are

$$\frac{\partial \pi_i}{\partial x_i} = A - 2x_i - (N-1)x_j + y_i + (N-1)y_j - 2dx_i = 0 \tag{9}$$

$$\frac{\partial \pi_i}{\partial y_i} = x_i - 2fy_i = 0 \tag{10}$$

Solving (10) for y_i^* yields to

$$y_i^* = \frac{1}{2f} x_i \tag{11}$$

Equation (11) shows that y_i^* depends only on the amount produced of the firm's private good and on the weight of the public good's cost function f. Therefore, the optimal level of y_i^* is independent of the production decision of the other firms and changes only if x_i^* changes. We summarize this observation in the following proposition.

Proposition 1

Each firm produces the private good x_i^* and the public good y_i^* in the same ratio which is determined by the weight of the public good cost function on the firm's profit f.

$$y_i^* = \frac{1}{2f} x_i$$

The intuition for this is as follows: For a firm the production of a public good has the only effect of increasing the price of the private good. In our case the effect of y_i on p is always constant and equal to 1. The marginal revenue of y_i is therefore x_i. The marginal costs are $2fy_i$. In the optimum the marginal revenue must equal the marginal costs. Therefore, we can see that the relationship between x_i and y_i is linear and the constant slope depends only on the weight of the public good's production cost f.

Next we want to determine the optimal amount of the private good for a firm. Solving (9) for x_i^* and using (11) and symmetry yields to

$$x_i^* = \frac{A}{1 + N - \frac{1}{2f}N + 2d} \tag{12}$$

To get economically reasonable values we restrict the possible values of the numbers of firms N in such a way that x_i^* cannot be negative which can only be the case if the weight of the public good's cost function is low ($f < 0.5$). In this case we restrict N to be smaller than $\overline{N}$ with

$$\overline{N} = \frac{2d + 1}{\frac{1}{2f} - 1} \tag{13}$$

For an optimum it is necessary and sufficient that the second-order conditions are fulfilled.

$$\frac{\partial^2 \pi}{\partial x^2} = -2 - 2d < 0 \tag{14}$$

$$\frac{\partial^2 \pi}{\partial y^2} = -2f < 0 \tag{15}$$

$$[\frac{\partial^2 \pi}{\partial x \partial y}]^2 = 1 < (-2-2d)(-2f) \tag{16}$$

After deriving the firms' optimal production decisions we want to look how these change due to market entry.

Therefore, suppose that an additional firm enters the market. With symmetry the new firm can use the same production technology as the incumbents. The new firm is going to produce the private and the public good. This has two effects: On the one hand competition increases due to the additional production of the private good which may lower the incentives to produce x_i. On the other hand the entrant's production of the public good raises the consumers' valuation of the private good and therefore gives an incentive to increase the production of the private good. Hence, it is not obvious in which direction an incumbent is going to adjust its production of the private good.

Proposition 2

If the number of competing firms N increases then

- *each incumbent reduces his production of the private good x_i^* if the weight of the public good's cost function on the profit ensures that the entrant produces more of the private good than of the public good ($f > 0.5$)*
- *each incumbent does not change his production of the private good x_i^* if the weight of the public good's cost function ensures that the entrant produces as much of the public good as of the private good ($f = 0.5$)*
- *each incumbent increases his production of the private good x_i^* if the weight of the public good's cost function on the profit ensures that the*

entrant produces more of the public good than of the private good ($f < 0.5$)

<u>Proof:</u>

To proof Proposition 2 we take the first derivative of x_i^* with respect to N.

$$\frac{\partial x}{\partial N} = -\frac{A}{[1 + N - \frac{1}{2f}N + 2d]^2} * (1 - \frac{1}{2f}) \tag{17}$$

We can see that the sign is negative if $f > 0.5$, is zero if $f = 0.5$ and is positive if $f < 0.5$.

Q.E.D.

Proposition 2 states the somehow surprising result that the effect on x_i^* depends only on the weight of the public good's cost function. This issue becomes clear if we consider the effects of a new firm on the incumbents. In the equilibrium the new firm produces the same amount of the public and of the private good as every incumbent after the adjustment process. Due to the chosen demand function, x and y influence the price p always with the same weight. Therefore, if in the equilibrium the additional competitor produces the same amount of the private good as of the public good, the firm does not influence the behavior of the incumbents. This is the case because the entrant does not alter the price and therefore in some way the incumbents do not even notice the market entry.

If the weight of the public good's cost function is small ($f < 0.5$) then every firm produces more of the public good than of the private good. This increases the price. Without altering their production the incumbents get a higher price for their last produced unit of the private good. Since the costs for the production did not change the incumbents should expand their production so that in the end the usual condition for an optimum "marginal revenue equals marginal cost" is fulfilled. If the weight of the public good's cost function is high ($f > 0.5$) then the opposite is true.

Using Proposition 1 and 2 we can can determine the shift of the public good production of a firm y_i.

Proposition 3

If the number of competing firms N increases then

- *each incumbent reduces his production of the public good y_i^* if due to the market entry it lowers its production of the private good*
- *each incumbent does not change his production of the public good y_i^* if due to the market entry it does not change its production of the private good*
- *each incumbent increases his production of the public good y_i^* if due to the market entry it raises its production of the private good*

Proposition 2 states under which circumstances every firm increases or decreases its production of the public good or holds it constant. Due to Equation (11) we know that for every firm the optimal amount of the public good is determined through its production of the private good because the production of the public good has just the effect of raising the achievable price for the private good. Therefore, it is straightforward that the adjustment of the amount of the public good gets in the same direction as the adjustment of the production of the private good. This is the case because due to higher amount produced of the private good the marginal revenue of the public good increases. Especially, this is true since the effect of the public good on the price of the private good is always constant and therefore independent of a change in the total produced amount of the private and of the public good.

Next we want to look at the variation of the total amount of the private and public good. By Proposition 2 and by Proposition 3 we know that under some circumstances every firm decreases its production of the private and the public good. Therefore it is questionable if the production of the entrant can compensate this decline of the old producers' production.

Proposition 4

Market entry unambiguously increases the total amount of the private good X and the public good Y.

Proof:

see Appendix

Therefore, we see that the production of the additional firm is always big enough to compensate for the loss of production by the incumbents. This implies an interesting consequence:
The outcome of a normal Cournot-Game with the number of firms progressing towards infinity is equivalent to the outcome of a game with perfect competition where the firms behave as price takers.
In our setup the production of the public good should break down if the firms behave as price takers because the motivation for the production of the public good is the change of the private good's price. Hence, in our model the market equilibrium of a Cournot-Game with a infinite number of firms is no longer equivalent to the market equilibrium where the firms behave as price taker because the equilibria are different.

With the knowledge about the change of the total amount of the private and the public good we can determine the variation of the price.

Proposition 5

If the number of competing firms N increases then

- *the price p decreases if every firm produces more of the private good than of the public good ($f > 0.5$)*
- *the price p does not change if every firm produces as much of the private good as of the public good ($f = 0.5$)*
- *the price p increases if every firm produces more of the public good than of the private good ($f < 0.5$)*

Proof:

see Appendix

The result gets intuitively clear if one takes into account the total change of the amount produced. We know by Proposition 1 that the ratio between the public and private good is always constant. This means that the absolute difference between the total amount of the private and public good is increasing with the number of active firms because the total production of both goods is rising when more firms produce. Therefore, if for example every firm produces more of the private good than of the public good market entry leads to a price reduction because every unit of the private and every unit of the public good influences the price with the same weight and the difference between the total produced amount of both goods increases. Only in the case where every firm produces as much of the private good as of the public good the price does not change due to market entry since the negative effect on the price through the public good is fully compensated through the positive effect on the price through the private good.

After analyzing the change of the amounts produced and the price we now want to look at the profits of the firms and the social surplus.

In a normal Cournot-Competition, i.e. without a public good and only with a private good, the incumbents dislike market entry since the entrant has a negative pecuniary externality through the additional supply of the private good on the already producing firms. Now in our setup the entrant may have also a positive pecuniary externality since the entrant also contributes to the public good.

Proposition 6

If the number of competing firms N increases then

- *the profit of the incumbents π_i decreases if every firm produces more of the private good than of the public good ($f > 0.5$)*
- *the profit of the incumbents π_i does not change if every firm produces as much of the private good as of the public good ($f = 0.5$)*
- *the profit of the incumbents π_i increases if every firm produces more of the public good than of the private good ($f < 0.5$)*

Proof:

see Appendix

By Proposition 2 and by Proposition 3 we know that with market entry and a high weight of the public good's cost function ($f > 0.5$) every firm produces less of both goods and that the price in the market decreases. Hence, it is straightforward that under these circumstances, the profit of the incumbents decrease, which is the usual effect of stronger competition. If $f < 0.5$, we get the surprising result that the incumbents prefer more competition. This is due to the fact that a symmetric competitor does not only produce the private good which has a pecuniary negative external effect on the incumbents, but also produces some amount of the public good and therefore has a pecuniary positive external effect on the incumbents. If the weight of the production cost of the public good is small, then the positive external effect of a market entry dominates over the negative effect.

One might think that the social surplus reacts ambiguously to a market entry because sometimes the firms get higher profits and the price for the private good rises. But this is not the case.

Proposition 7

If the number of competing firms N increases then the social surplus always gets higher.

Proof:

see Appendix

If we just look at the consumer surplus we see that it is increasing with the number of active firms. This can be explained by two effects. Firstly, due to market entry the production of the public good increases. This has the effect that every consumer values the private good higher which has a positive effect on the consumer surplus. Secondly, the market entry leads to a tougher competition in the proprietary sector which increases the total production and has once again a positive effect on the consumer surplus. So even if the market entry leads to higher prices we get a higher consumer surplus. In our setup the relevant issue is the total produced amount of both goods. Since the firms cannot make any price discrimination a higher produced amount of the goods lead to a higher consumer surplus.

Proposition 6 shows that with a low weight of the public good's production cost market entry increases the profits of the firms. In this case it is obvious that the total surplus also increases. Nevertheless, if the firms' profit decrease through the market entry the gain in the consumer surplus compensates this effect and results in a higher total surplus. Therefore, a social planer always prefers market entry.

4 The General Case

In this section we want to analyze the general adjustment process of an incumbent if the other firms' total production changes. We therefore use a general demand function.

Optimal Production Plan

Profit maximization, given the production of all other companies, yields the following first order conditions for firm i

$$\frac{\partial \pi_i}{\partial x_i} = p + x_i \frac{\partial p}{\partial x_i} - \frac{\partial K_i(x_i)}{\partial x_i} = 0 \tag{18}$$

$$\frac{\partial \pi_i}{\partial y_i} = x_i \frac{\partial p}{\partial y_i} - \frac{\partial K_i(y_i)}{\partial y_i} = 0 \tag{19}$$

Equation (18) displays the standard condition for the optimal production of the private good where marginal costs equal marginal revenue. Equation (19) shows that the production of the public good has only an indirect effect on the profit. By raising the amount of y, the consumers' willingness to pay increases, and therefore the firm achieves a higher price for their produced private goods.

In order for (18) and (19) to describe a maximum, the following second-order conditions have to be fulfilled.

$$2 * \frac{\partial p}{\partial x_i} + x_i \frac{\partial^2 p}{\partial x_i^2} - \frac{\partial^2 K_i(x_i)}{\partial x_i^2} < 0 \tag{20}$$

$$x_i \frac{\partial^2 p}{\partial y_i^2} - \frac{\partial^2 K(y_i)}{\partial y_i^2} < 0 \tag{21}$$

$$[\frac{\partial p}{\partial y_i} + x_i \frac{\partial^2 p}{\partial x_i \partial y_i}]^2 < [2 * \frac{\partial p}{\partial x_i} + x_i \frac{\partial^2 p}{\partial x_i^2} - \frac{\partial^2 K_i(x_i)}{\partial x_i^2}] * [x_i \frac{\partial^2 p}{\partial y_i^2} - \frac{\partial^2 K(y_i)}{\partial y_i^2}] \tag{22}$$

The Adjustment Process with Direct and Indirect Effects

Suppose that firm i faces a fixed exogenous change in the produced amount of X_{-i} and/ or Y_{-i}. Firm i's optimal adjustment is determined by totally differentiating equations (18) and (19).

Total derivative of $\frac{\partial \pi_i}{\partial x_i}$:

$$[2 * \frac{\partial p}{\partial x_i} + x_i \frac{\partial^2 p}{\partial x_i^2} - \frac{\partial^2 K(x_i)}{\partial x_i^2}]dx_i + [\frac{\partial p}{\partial y_i} + x_i \frac{\partial^2 p}{\partial x_i \partial y_i}]dy_i \tag{23}$$
$$+[\frac{\partial p}{\partial X_{-i}} + x_i \frac{\partial^2 p}{\partial x_i \partial X_{-i}}]dX_{-i} + [\frac{\partial p}{\partial Y_{-i}} + x_i \frac{\partial^2 p}{\partial x_i \partial Y_{-i}}]dY_{-i} = 0$$

Total derivative of $\frac{\partial \pi_i}{\partial y_i}$:

$$[\frac{\partial p}{\partial y_i} + x_i \frac{\partial^2 p}{\partial x_i \partial y_i}]dx_i + [x_i \frac{\partial^2 p}{\partial y_i^2} - \frac{\partial^2 K_i(y_i)}{\partial y_i^2}]dy_i \tag{24}$$
$$+[x_i \frac{\partial^2 p}{\partial y_i \partial X_{-i}}]dX_{-i} + [x_i \frac{\partial^2 p}{\partial y_i \partial Y_{-i}}]dY_{-i} = 0$$

Equation (23) and (24) show that we can decompose the overall effect into direct and indirect effects in order to understand the complex adjustment process.
1. The direct effect influences x_i (y_i) through a change in X_{-i} or Y_{-i} without depending on a change in the corresponding complement y_i (x_i).
2. The indirect effect influences the optimal level of x_i (y_i) through a change in the corresponding complement y_i (x_i).

In the next two sections, we want to analyze the adjustment process of the private and the public good and show what determines the sign of the direct and indirect effects.

The Adjustment Process of y_i

By Equation (19) we know that the positive effect of the production of y_i is determined through the positive impact on p and weighted by the produced amount of x_i. Therefore, there are two possibilities for X_{-i} and Y_{-i} to influence the optimal produced amount of y_i.
First, it can change the influence of y_i on p (the direct effect). When the relevant second-order derivative is zero, this direct effect does not exist.
But there is a second possibility to influence the optimal amount of y_i (the indirect effect). It may be the case that because of the change in X_{-i} and Y_{-i}, the firm changes its production of x_i. If x_i changes, then the firm should adjust its production of y_i. Once again, this adjustment can depend on a second-order derivative, namely on the the sign of the cross-derivative weighted by x_i. If the cross-derivative is positive or zero, it is obvious that an increase in x_i leads to an increase in y_i. This result is very intuitive. Increasing y_i raises the price p and therefore the firm receives more money for every sold unit of x_i. This gets more attractive if x_i is higher compared to the situation before.
But we have to keep in mind the size of the impact of y_i on p. Because of the higher x_i the impact may decrease meaning that the cross-derivative is negative and therefore the incentive to increase y_i is lower. It might be the case that this effect dominates and therefore an increase of x_i leads to a reduction of y_i. Intuitively, this means that the increase in x_i influences the impact of y_i on p in such a negative way that, although an increasing p acts on more x_i, it makes no sense to keep up the old level of y_i because the small resulting change in p does not justify the costs of y_i. We summarize these findings in the first two propositions.

Proposition 8

(a) A change in X_{-i} has a direct effect on the production level of y_i only if $\frac{\partial p}{\partial y \partial x}$ is not zero.
An increase in X_{-i} has a positive direct effect on the production level of y_i if

$\frac{\partial p}{\partial y \partial x} > 0$ *and a negative direct effect if* $\frac{\partial p}{\partial y \partial x} < 0$.

(b) A change in Y_{-i} *has a direct effect on the production level of* y_i *only if* $\frac{\partial p}{\partial y^2}$ *is not zero.*
An increase in Y_{-i} *has a positive direct effect on the production level of* y_i *if* $\frac{\partial p^2}{\partial y^2} > 0$ *and a negative direct effect if* $\frac{\partial p}{\partial y^2} < 0$.

Proof:
To isolate the direct effect on y_i through X_{-i} we take the total derivative of $\frac{\partial \pi_i}{\partial y_i}$ and set $dx_i = dY_{-i} = 0$:

$$[\overbrace{x_i \frac{\partial^2 p}{\partial y_i^2} - \frac{\partial^2 K_(y_i)}{\partial y_i^2}}^{\text{- (SOC)}}]dy_i + [\overbrace{x_i \frac{\partial^2 p}{\partial y_i \partial X_{-i}}}^{?}]dX_{-i} = 0 \tag{25}$$

Equation (25) shows that an increase in y_i always decreases the left hand side of the equation because the expression in brackets is the second-order condition with respect to y_i. By Equation (21) this must be negative.
Therefore, if dX_{-i} is positive dy_i must be

- equal to zero if the cross-derivative of p is zero
- positive if the cross-derivative of p is positive
- negative if the cross-derivative of p is negative

To isolate the direct effect on y_i through Y_{-i} we take the total derivative of $\frac{\partial \pi_i}{\partial y_i}$ and set $dx_i = dX_{-i} = 0$:

$$[\overbrace{x_i \frac{\partial^2 p}{\partial y_i^2} - \frac{\partial^2 K_(y_i)}{\partial y_i^2}}^{\text{- (SOC)}}]dy_i + [\overbrace{x_i \frac{\partial^2 p}{\partial y_i \partial Y_{-i}}}^{?}]dY_{-i} = 0 \tag{26}$$

Equation (26) shows that an increase in y_i always decreases the left hand side because the expression in brackets is the second-order condition with respect to y_i. By Equation (21) this must be negative.
Therefore, if dY_{-i} is positive, dy_i must be

- equal to zero if the second-derivative of p with respect to y is zero
- positive if the second-derivative of p with respect to y is positive
- negative if the second-derivative of p with respect to y is negative

Q.E.D.

Proposition 9

A change in X_{-i} and/ or Y_{-i} can have an indirect effect on the optimal level of y_i through changing x_i. If the cross-derivative is positive or zero this direct effect is always positive. If the cross-derivative is negative the effect is ambiguous.

Proof:
To isolate the indirect effect through x_i on y_i we take the total derivative of $\frac{\partial \pi_i}{\partial y_i}$ and set $dX_{-i} = dY_{-i} = 0$:

$$[\overbrace{\frac{\partial p}{\partial y_i}}^{+} + \overbrace{x_i \frac{\partial^2 p}{\partial x_i \partial y_i}}^{?}]dx_i + [\overbrace{x_i \frac{\partial^2 p}{\partial y_i^2} - \frac{\partial^2 K_(y_i)}{\partial y_i^2}}^{\text{- (SOC)}}]dy_i = 0 \tag{27}$$

From (27) we can see that the effect of dx_i is only clear if the cross-derivative is positive or zero.

Q.E.D.

Adjustment Process of x_i

From Equation (23) it is obvious that, even when the second-order derivatives are zero there is a direct effect of X_{-i} and Y_{-i} on x_i. The interpretation is that a rise in X_{-i} (Y_{-i}) decreases (increases) the price and therefore gives an incentive to lower (rise) the production of x_i. With the second-order derivatives different from zero, the direct effect is no longer straightforward

because there is a change of the impact of x_i on the price p. Suppose, for illustration, that the second derivative of p with respect to x is positive. Then it might be the case that an increase in X_{-i} leads to an increase in x_i. Hence even though by raising X_{-i} the price p falls, it is optimal to increase the production of x_i. This is explained by paying attention to the effect of x_i on p. Through raising X_{-i} one arrives at a point on the demand curve where the price ceases to react very strongly to a change of x_i. Compared to the situation before, this gives an incentive to increase the production of x_i. Taken the two forces together it may be that a higher production level of x_i will be optimal.
As in the case of y_i there is a feedback from the first-order condition of π_i with respect to y_i and therefore an indirect effect appears. If the cross-derivative is zero then a raise in y_i leads to an increase in x_i because a higher y_i has a positive impact on the price. But once again we have to keep in mind the cross-derivative. If it is negative, meaning that through an increase of y_i the price reacts more sensitive to a change of x_i, then the negative effect (loosing money on all old units of x_i) of an increase in x_i gets stronger. This can lead to a reduction of x_i if y_i increases. We summarize the findings in the two propositions below.

Proposition 10

(a) The direct effect of a change in X_{-i} on x_i is driven by two forces:

- *Firstly, an increase of X_{-i} decreases the price and therefore gives an incentive to reduce the production of x_i.*
- *Secondly, an increase of X_{-i} might alter the reaction of the price p with respect to x_i, depending on the sign of the second derivative of p with respect to x. If the sign is zero or negative the whole direct effect is negative. If the sign is positive then the whole effect is ambiguous.*

(b) The direct effect of a change in Y_{-i} on x_i is driven by two forces:

- *Firstly, an increase of Y_{-i} increases the price and therefore gives an incentive to raise the production of x_i.*
- *Secondly, an increase of Y_{-i} might alter the reaction of the price p with respect to x, depending on the sign of the cross derivative of p. If the sign is zero or positive the whole direct effect is positive. If the sign is negative then the whole effect is ambiguous.*

<u>Proof:</u>

To get the direct effect through X_{-i} on x_i we take the total differentiation of $\frac{\partial \pi_i}{\partial x_i}$ and set $dy_i = dY_{-i} = 0$:

$$[\overbrace{2 * \frac{\partial p}{\partial x_i} + x_i \frac{\partial^2 p}{\partial x_i^2} - \frac{\partial^2 K(x_i)}{\partial x_i^2}}^{\text{- (SOC)}}]dx_i + [\overbrace{\frac{\partial p}{\partial X_{-i}}}^{-} + \overbrace{x_i \frac{\partial^2 p}{\partial x_i \partial X_{-i}}}^{?}]dX_{-i} = 0 \quad (28)$$

Equation (28) shows that an increase in X_{-i} decreases the left hand side if the second derivative of p with respect to x is zero or negative. If the derivative is positive it can happen that an increase of X_{-i} raises the value of the left hand side and therefore x_i must increase so that the equation holds true.

To get the direct effect through Y_{-i} on x_i we take the total differentiation of $\frac{\partial \pi_i}{\partial x_i}$ and set $dy_i = dX_{-i} = 0$:

$$[\overbrace{2 * \frac{\partial p}{\partial x_i} + x_i \frac{\partial^2 p}{\partial x_i^2} - \frac{\partial^2 K(x_i)}{\partial x_i^2}}^{\text{- (SOC)}}]dx_i + [\overbrace{\frac{\partial p}{\partial Y_{-i}}}^{+} + \overbrace{x_i \frac{\partial^2 p}{\partial x_i \partial Y_{-i}}}^{?}]dY_{-i} = 0 \quad (29)$$

Equation (29) shows that an increase in Y_{-i} increases the left hand side if the cross derivative of p is zero or positive. If the derivative is negative it can happen that an increase of Y_{-i} decreases the value of the left hand side and therefore x_i must decrease so that the equation holds true.

Q.E.D.

Proposition 11

A change in Y_{-i} and/ or X_{-i} can have an indirect effect on the optimal production level of x_i through changing y_i. If the cross-derivative is positive or zero this indirect effect is always positive. If the cross-derivative is negative the effect is ambiguous.

Proof:
To get the indirect effect on x_i through y_i we take the total differentiation of $\frac{\partial \pi_i}{\partial x_i}$ and set $dX_{-i} = dY_{-i} = 0$:

$$[\overbrace{2 * \frac{\partial p}{\partial x_i} + x_i \frac{\partial^2 p}{\partial x_i^2} - \frac{\partial^2 K(x_i)}{\partial x_i^2}}^{\text{- (SOC)}}]dx_i + [\overbrace{\frac{\partial p}{\partial y_i}}^{+} + \overbrace{x_i \frac{\partial^2 p}{\partial x_i \partial y_i}}^{?}]dy_i = 0 \tag{30}$$

From (30) we can see that the effect of dy_i is only clear if the cross-derivative is positive or zero.

Q.E.D.

In the standard textbook Cournot Competition, i.e there is only a private good and no public good, the adjustment is determined only through the direct effect of dX_{-i}. Please note that even in the textbook case this direct effect, as stated in Proposition 10, can be negative, zero or positive.

As seen, the introduction of the public good adds a number of additional effects.

1. In addition to the direct effect of dX_{-i} on x_i there is a direct effect of dY_{-i} on x_i which also influences the price and the impact of x_i on p and therefore gives an incentive to adjust x_i.

2. Furthermore two direct effects act on y_i. These direct effects do not work over influencing the price directly, they just work over altering the impact of y_i on p and therefore give an incentive to adjust y_i.

3. If there is a direct effect on the private or public good than there begins a feedback loop due to the indirect effects. Consider for example that there is only a direct effect on x_i and therefore the optimal amount of x_i changes. Then this has a feedback effect on the optimal amount of y_i, because y_i depends on the produced amount of x_i. The resulting change in y_i once again has a feedback effect on x_i, because a change of y_i alters the price and perhaps the impact of x_i on p and therefore gives an incentive to adjust the production of x_i and so on.

Summarizing Proposition 8-11 we see that there can be direct and indirect effects on x_i and y_i through a change in X_{-i} and Y_{-i}. Especially, the two indirect effects influence each other and make the whole adjustment process very complex.

5 Conclusion

In this paper we have shown that the adjustment process of a firm's optimal production plan can be analyzed by examining direct and indirect effects. Furthermore, we have seen that the Cournot-Nash Equilibrium with an increasing number of firms does not converge to the market equilibrium where the firms act as price-takers. A social planer always prefers market entry but the effect on the profits of the incumbents is ambiguous. This brings us to the interesting fact that in some cases firms should try to encourage market entry e.g. through lowering market entry-barriers.

Appendix

Proof of Proposition 4:

For the first part of the proof we take the derivative of X with respect to N and look at the sign.

$$\frac{\partial X}{\partial N} = \frac{\partial [N * x(N)]}{\partial N} = N * \frac{\partial x}{\partial N} + x \tag{31}$$

$$\frac{\partial X}{\partial N} = -N * \frac{A}{[1 + N - \frac{1}{2f}N + 2d]^2} * (1 - \frac{1}{2f}) + \frac{A}{1 + N - \frac{1}{2f}N + 2d} \tag{32}$$

$$\frac{\partial X}{\partial N} = \frac{A}{1 + N - \frac{1}{2f}N + 2d}(-N * \frac{1}{1 + N - \frac{1}{2f}N + 2d} * (1 - \frac{1}{2f}) + 1) \tag{33}$$

$$\frac{\partial X}{\partial N} = \frac{A * [1 + 2d]}{(1 + N - \frac{1}{2f}N + 2d)^2} > 0 \tag{34}$$

By Proposition 1 we know that the private and the public good are always individually produced in the same ratio. Summing up, we see that the total amounts produced must also have the same ratio. If X is always increasing, then Y is also always increasing.

Q.E.D.

Proof of Proposition 5:

For the proof we use the fact that $y = \frac{1}{2f}x$ (11) and rewrite the demand function (5) as follows:

$$p = A - X + Y = A - X(N) + \frac{1}{2f}X(N) = A + X(N)(-1 + \frac{1}{2f}) \tag{35}$$

$$p = A + N * \frac{A}{a + N - \frac{1}{2f}N + 2d}(-1 + \frac{1}{2f}) \tag{36}$$

Now we can take the first derivative of p with respect to N and look at the sign.

$$\frac{\partial p}{\partial N} = (- 1 + \frac{1}{2f}) * \frac{A(1 + 2d)}{[1 + N - \frac{1}{2f}N + 2d]^2} \tag{37}$$

Therefore

- $\frac{\partial p}{\partial N} < 0$ if $f > 0.5$
- $\frac{\partial p}{\partial N} = 0$ if $f = 0.5 \rightarrow$ p=A
- $\frac{\partial p}{\partial N} > 0$ if $f < 0.5$

Q.E.D.

Proof of Proposition 6:

$$\pi_i = p * x_i - dx^2 - f[\frac{1}{2f}x_i]^2 = p * x_i - x_i^2(d + \frac{1}{4f}) \tag{38}$$

$$\pi_i = (A + N * \frac{A(-1 + \frac{1}{2f})}{1 + N - \frac{1}{2f}N + 2d}) * \frac{A}{1 + N - \frac{1}{2f}N + 2d} - (\frac{A}{1 + N - \frac{1}{2f}N + 2d})^2(d + \frac{1}{4f}) \tag{39}$$

$$\pi_i = \frac{A^2(1 + d - \frac{1}{4f})}{(1 + N - \frac{1}{2f}N + 2d)^2} \tag{40}$$

$$\frac{\partial \pi_i}{\partial N} = 2A^2 * \frac{1}{(1 + N - \frac{1}{2f}N + 2d)^3}(1 - \frac{1}{2f})(-1 - d + \frac{1}{4f}) \tag{41}$$

The last term $-1 - d + \frac{1}{4f}$ is always negative because of the SOC in the maximization problem. We know by Equation (22) that it must be true that

$$1^2 < (-2 - 2d)(-2f) \tag{42}$$

Solving for f yields to

$$f > \frac{1}{4 + 4d} \tag{43}$$

Assuming $-1 - d + \frac{1}{4f} < 0$ and solving for f leads to:

$$f > \frac{1}{4 + 4d} \tag{44}$$

Therefore we have to analyze the first and the second term.

Case 1: $f = 0.5$

Then $1 - \frac{1}{2f} = 0$ and therefore $\frac{\partial \pi}{\partial N} = 0$.

Case 2: $f > 0.5$

Then $1 - \frac{1}{2f} > 0$ and the sign of $\frac{\partial \pi}{\partial N}$ depends on N because of the term $(1 + N - \frac{1}{2f}N + 2d)$.

This term is always positive:

$$1 + N - \frac{1}{2f}N + 2d = 1 + N(1 - \frac{1}{2f}) + 2d \tag{45}$$

Therefore $\frac{\partial \pi}{\partial N} < 0$ for all $N > 0$

Case 3: $f < 0.5$

Then $1 - \frac{1}{2f} < 0$ and the sign of $\frac{\partial \pi}{\partial N}$ depends on N because of the term $(1 + N - \frac{1}{2f}N + 2d)$.

This term is zero if

$$N^* = \frac{2d+1}{\frac{1}{2f}-1} > 0 \tag{46}$$

The slope of $(1 + N - \frac{1}{2f}N + 2d)$ with respect to N is

$$\frac{\partial(1 + N - \frac{1}{2f}N + 2d)}{\partial N} = 1 - \frac{1}{2f} < 0 \tag{47}$$

Therefore $\frac{\partial \pi}{\partial N} > 0$ in the relevant area where $N < \overline{N}$.

Q.E.D.

Proof of Proposition 7:

First we calculate the effect of more competition on the consumer surplus.

$$CS = (A + Y - p) * X * 0.5 = 0.5 * X^2 \tag{48}$$

We see that the consumer surplus always increases. In the case of $f \geq 0.5$, this is quite obvious. The total sold amount of the private good increases and the price does not raise. If the weight of the public good cost function is low, than the price for the private good increases and the total production of both goods also increases. Nevertheless the consumers benefit from more competition, because due to the higher production of the public good, their willingness to pay increases and therefore their surplus increases even though the price raises.

The total surplus is the sum of the companies profits and the consumer surplus.

$$TS = N * \pi + CS \tag{49}$$

To see the reaction of the total surplus caused by a variation in N, we take the first derivative of TS with respect to N and look at the sign.

$$\frac{\partial TS}{\partial N} = \frac{\partial(N * \pi)}{\partial N} + \frac{\partial CS}{\partial N} \tag{50}$$

$$\pi * N = N * \frac{A^2(1 + d - \frac{1}{4f})}{(1 + N - \frac{1}{2f} * N + 2d)^2} \tag{51}$$

$$CS = \frac{1}{2} * \frac{(AN)^2}{(1 + N - \frac{1}{2f} * N + 2d)^2} \tag{52}$$

$$\frac{\partial TS}{\partial N} = \frac{A^2(1 + d - \frac{1}{4f})(1 - N + \frac{1}{2f}N + 2d)}{(1 + N - \frac{1}{2f}N + 2d)^3} + \frac{A^2 N(1 + 2d)}{(1 + N - \frac{1}{2f}N + 2d)^3} \tag{53}$$

$$\frac{\partial TS}{\partial N} = \frac{A^2}{(1 + N - \frac{1}{2f}N + 2d)^3}[(1 + d - \frac{1}{4f})(1 - N + \frac{1}{2f}N + 2d) + N(1 + 2d)] \tag{54}$$

$$\frac{\partial TS}{\partial N} = \frac{A^2(1 + d - \frac{1}{4f})(1 + 2d)}{(1 + N - \frac{1}{2f}N + 2d)^3} + \frac{N(d + \frac{1}{4f} + \frac{1}{2f} + \frac{d}{2f} - \frac{1}{8f^2})}{(1 + N - \frac{1}{2f}N + 2d)^3} \tag{55}$$

If $f \geq 0.5$ then the denominator is always positive. As stated in the proof of Proposition 6, expression $(1 + d - \frac{1}{4f})$ is always positive. Therefore, the first term is positive as well. The second term is also positive because $\frac{1}{2f} > \frac{1}{8f^2}\ \forall\ f \geq 0.5$.
If $f < 0.5$ then we know that π and CS are increasing and therefore the social surplus must also increase.

Q.E.D.

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