From Preferences to Choice: a Completion Approach *

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Abstract

In an Anscombe and Aumann (1963) [1] setting, we generalize the model of Gilboa, Maccheroni, Marinacci and Schmeidler (2010) [16] by studying the problem of a Decision Maker that considers several potential completion criteria in order to complete an "objective" incomplete preference relation. We show how the attitude of the decision maker toward the potential completion criteria influences the final aggregation process.

1 Introduction

Since the seminal paper by Aumann (1962) [2], modelling incomplete preferences has become a standard problem in the field of decision theory and several authors, Aumann (1962) [2], Bewley (2002) [3], Shapley and Baucells (1998) [25] and Dubra, Maccheroni and Ok (2004) [12] among others, found a way of representing this type of preferences either in an ambiguity setup or in a risk based framework.

An "apparently" independent problem in decision theory under uncertainty has been the resolution of Ellsberg type paradoxes like the one of Ellsberg (1961) [13]. Some of the most satisfactory and elegant resolutions of this latter problem were proposed by Schmeidler (1989) [23] and by Gilboa and Schmeidler (1989) [17] (GS hereafter). The paper by Gilboa, Maccheroni, Marinacci and Schmeidler (2010) [16] (GMMS hereafter) provides a bridge between these two type of issues by finding a way of completing an "objective" incomplete preference relation in such a way that the resulting complete "subjective" preference relation has a maxmin functional representation a là GS [17]. The key axiom behind the GMMS [16]'s representation is an axiom called "caution" that describes the behavior of a particularly prudent Decision Maker (DM hereafter) that prefers constant acts whenever two acts are not comparable in an "objective" sense.

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Our paper wants to understand how an agent can complete in a consistent way an initial incomplete piece of information and it can be seen as a generalization of GMMS [16]'s framework. In particular we model the problem of a DM that has to complete an incomplete preference relation and in order to achieve her objective the DM considers a finite number of potential completion criteria. Since a choice has to be performed the agent aggregates these potential completion criteria in a finally implemented choice correspondence that represents the choices that will be eventually made[1].

Our framework has an interesting interpretation from a statistical decision theoretic viewpoint. There is in fact a tight link between robust statistics and decision making under ambiguity[2]. In particular, following the suggestion provided by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2013a) [7], we can interpret the "objective" incomplete preference relation $\succsim^*$ in terms of a datum of the problem that has to be inserted inside the DM's subjective framework. The "objective" information is commonly accepted [3] and, as a consequence, all the potential completion criteria have to be consistent with this incomplete preference relation. It is as if the DM knows that the correct model belongs to the class of models consistent with the "objective" incomplete preference relation. Hence she has to choose a subset of models from this class.

We now provide a simple clarifying example.

Example 1 A DM considers 3 potential completion criteria, 2 of which have a representation a là GS [17] and the remaining one has an invariant biseparable preference a là Ghirardato, Maccheroni and Marinacci (2004) [15] (GMM hereafter). All 3 potential completion criteria have to be consistent with $\succsim^*$ and they are aggregated by the agent in the"subjective" choice correspondence C^o. The situation just described is represented in Figure 1.

▲

[1] The fact that the DM considers only a finite number of criteria can be interpreted in terms of bounded rationality. Our agent is computationally bounded and it is not able to contemplate an infinite number of potential completion criteria.

[2] For a complete discussion and proper formalization of the topic see Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2013b) [6].

[3] According to the classical interpretation provided by GMMS [16], it is as if the DM has a proof of the correctness of her "objective" decisions.

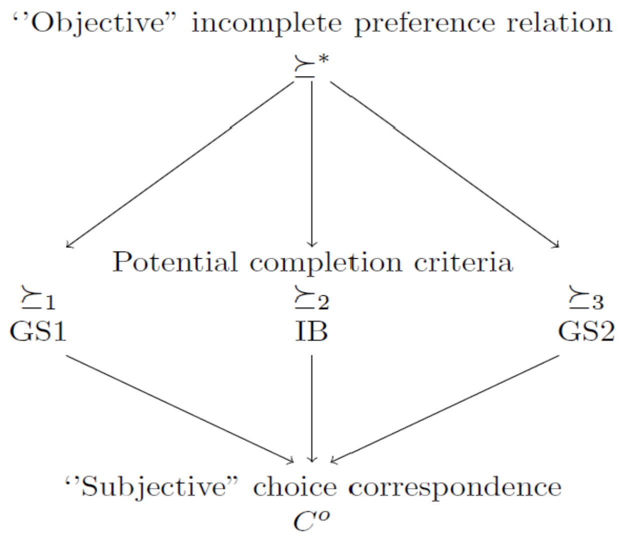

Figure 1

The attitude that the DM has toward the potential completion criteria plays a leading role in our paper. A first basic Harsanyi type result is introduced in Proposition 16 in which we show how consistency of the final aggregator w.r.t. the completion criteria it only implies that the final functional representation is a (not unique) combination of the criteria.

The main representation results of the paper are characterized by a negative attitude of the agent toward the potential completion criteria. This pessimistic behavior is axiomatically formalized by Criteria Uncertainty Aversion Axiom 17. In Proposition 18 and Proposition 22 and corollary we show that only a really "cautious" DM will end up in a representation a là GMMS [16] and that in general a distrustful agent is characterized by the following criteria:

$$C^o(A) = \underset{f \in A}{argmax} \left\{ \underset{\gamma \in \Gamma}{min} \sum_{j=1}^{N} \gamma_j \cdot I_j(f) \right\}$$

where $\{I_i : B_0(\Sigma) \to \mathbb{R}\}_{i=1}^{N}$ are monotonic, constant additive and positively homogenous linear functionals and $\Gamma \subseteq \triangle(\{1, 2, \ldots, N\})$ is a closed and convex set.

In Theorem 25 we give a variational representation w.r.t. the potential completion criteria. The result is clearly inspired by the work of Maccheroni, Marinacci and Rustichini (2006) [20] (MMR hereafter). In this Theorem we consider an agent that satisfies Criteria Uncertainty Aversion Axiom but this time we introduce two new axioms, Volatility and the Criteria Betweeness, that allow the following niveloidal representation:

$$C^o(A) = \underset{f \in A}{argmax} \left\{ \underset{\gamma \in \triangle(\{1,2,\ldots,N\})}{min} \left(\sum_{j=1}^{N} \gamma_j \cdot I_j(u(f)) + c(\gamma) \right) \right\}$$

where $c : \triangle(\{1, 2, \ldots, N\}) \to [0, \infty]$ is a lower semicontinuous, grounded and convex function. Criteria Betweeness axiomatically captures the idea that the final valuation

$I_o(u(f))$ of a generic non constant act f has to be always between the valuations of 2 potential completion criteria, i.e. there always exist 2 criteria i and j such that $I_i(u(f)) \leq I_o(u(f)) \leq I_j(u(f))$. On the other side Volatility formalizes the idea that the potential completion criteria considered needs to have enough variability in their valuations in such a way that for any 2 non constant acts f, g there always exist a criteria i such that $| I_i(u(f)) - I_i(u(g)) | \geq | I_o(u(f)) - I_o(u(g)) |$.

The paper consists overall of 5 sections. In the "Framework" section we describe the setup and we introduce the preliminaries necessary to properly understand the model. In the sections "A simple aggregation process", "A pessimistic attitude", "A variational approach" we axiomatically characterize a DM that has several different attitudes toward the potential completion criteria. Finally we try to insert our paper inside the current literature in the "Related Literature" section. All the proofs are contained inside the "Appendix".

2 Framework

We make use of a version of the Anscombe and Aumann (1963) [1] model as restated by Fishburn (1970) [14]. A von-Neumann-Morgenstern lottery is a finite support probability distribution over the set of outcomes X. The set of lotteries L over X, is endowed with a mixing operation: for every P, $Q \in L$ and every $\alpha \in [0, 1]$, $\alpha \cdot P + (1 - \alpha) \cdot Q \in L$ is defined pointwise over X. The finite set of states of the world is S and it is endowed with an algebra Σ of events. The set of finitely additive probabilities on Σ is denoted as $\Delta(\Sigma)$ and it is endowed with the eventwise convergence topology. The set of simple acts $\mathcal{F}$ consists of all simple measurable functions $f : S \to L$ and it is convexified by performing a pointwise mixture operation on S. We denote with $\mathcal{F}_c$ the set of constant acts[4] and with $\Im$ the set of all non empty finite subsets of $\mathcal{F}$.

DM is characterized by a preference relation $\succeq^*$ that represents "objective rationality"[5] and she evaluates several "subjective" preference relations $\{\succeq_i\}_{i=1}^N$ representing potential completion criteria that she considers possible in order to complete the incomplete relation $\succeq^*$. The choice correspondence $C^o : \Im \to \Im$ represents the final choices made by DM and it is the result of the aggregation of the criteria $\{\succeq_i\}_{i=1}^N$.

3 A simple aggregation process

In this section we explore the implications of the simplest form of consistency that we can impose in order to link the potential completion criteria $\{\succeq_i\}_{i=1}^N$ with the choice correspondence C^o. We say that the incomplete preference relation $\succsim^*$ is Bewley if it satisfies the following set of axioms:

Axiom 2 *(Reflexivity) For any $f \in \mathcal{F}$ we have that $f \succeq^* f$;*

Axiom 3 *(Transitivity) If f, g, $h \in \mathcal{F}$ $f \succeq^* g$ and $g \succeq^* h$ then $f \succeq^* h$;*

[4] An act f is constant iff $\exists P \in L$ such that $\forall s \in S$ we have that $f(s) = P$.

[5] If $f \succsim^* g$, it is as if the DM has a correct proof of the fact that f is better than g.

Axiom 4 *(Nontriviality) There are* $f, g \in \mathcal{F}$ *such that* $f \succ^* g$*;*

Axiom 5 *(Monotonicity) For every* $f, g \in \mathcal{F}$*,* $f(s) \succeq^* g(s)$ *for all* $s \in S$ *implies* $f \succeq^* g$*;*

Axiom 6 *(Continuity) For all* $f,\ g,\ h, \in \mathcal{F}$ *the sets* $\{\lambda \in [0,\ 1] : \ \lambda \cdot f + (1-\lambda) \cdot g \succeq^* h\}$ *and*

$\{\lambda \in [0,\ 1] : \ h \succeq^* \lambda \cdot f + (1-\lambda) \cdot g\}$ are closed in $[0,\ 1]$;

Axiom 7 *(Independence) For every* $f, g, h \in \mathcal{F}$ *and* $\alpha \in (0,\ 1)$ *we have that* $f \succeq^* g$ *if and only if* $\alpha \cdot f + (1-\alpha) \cdot h \succeq^* \alpha \cdot g + (1-\alpha) \cdot h$*;*

Axiom 8 *(C-completeness) If* $\forall f,\ g, \in \mathcal{F}_c$ *either* $f \succeq^* g$ *or* $g \succeq^* f$*.*

On the other side we say that the set of potential completion criteria $\{\succeq_i\}_{i=1}^N$ and the choice correspondence C^o are Invariant Biseparable[6] if they satisfy Reflexivity Axiom 2, Transitivity Axiom 3, Nontriviality Axiom 4, Monotonicity Axiom 5, Continuity Axiom 6 and the following types of completeness and independence:

Axiom 9 *(Completeness) For all* $f,\ g \in \mathcal{F}$*, either* $f \succeq_i g$ *or* $g \succeq_i f$*;*

Axiom 10 *(C-Independence) For every* $f, g \in \mathcal{F}$*,* $h \in \mathcal{F}_c$ *and* $\alpha \in (0,\ 1)$ *we have that* $f \succeq_i g$ *if and only if* $\alpha \cdot f + (1-\alpha) \cdot h \succeq_i \alpha \cdot g + (1-\alpha) \cdot h$*;*

Each potential completion criterion $\succsim^i$ has to be a subrelation of the Bewlian preference relation $\succsim^*$. Hence we assume that each potential completion criteria $\succeq_i$is related to the incomplete preference relation $\succsim^*$ through the following consistency assumption:

Axiom 11 *(Consistency)* $f \succeq^* g$ *implies* $f \succeq_i g$*;*

The minimal consistency requirement that can be imposed in order to link the "subjective" choice correspondence $C^o : \Im \to \Im$ with the potential completion criteria $\{\succeq_i\}_{i=1}^N$ is described by the following axiom:

Axiom 12 *(Consistency Toward Criteria)* $f \succeq_i g$ *for* $i = 1, \ldots, N$ *implies that* $f \in C^o(\{f,\ g\})$*.*

The issue of this section is to understand which type of functional representation can be obtained by using this latter axiom. Before stating the first result of the paper we need to introduce the concept of joint convexity and unambiguous preference.

Definition 13 *We say that a finite set of preferences* $\{\succeq_i\}_{i=0}^N$ *over a set* $\mathcal{F}$ *satisfies **joint convexity** if each one of the preference relations admits a representation in terms of real valued functionals* $\{I_i\}_{i=0}^N$ *and the range of the functional* $I = (I_0, I_1, \ldots,\ I_N)$ *is a convex subset of* $\mathbb{R}^{N+1}$*.*

[6] In Appendix A we provide the axioms that characterize an Invariant Biseparable choice correspondence.

Definition 14 *We say that a finite set containing both preferences* $\{\succeq_i\}_{i=1}^N$ *over a set* $\mathcal{F}$ *and choice correspondences* $\{C^i\}_{i=1}^M$ *over the set of all non empty finite subsets of* $\mathcal{F}$ *satisfies* ***joint convexity*** *if the set obtained by putting together the preferences* $\{\succeq_i\}_{i=1}^N$ *and the revealed preferences associated to the choice correspondences* $\{C^i\}_{i=1}^M$ *is jointly convex.*

If all the potential completion criteria $\{\succeq_i\}_{i=1}^N$ are expected utilities then the assumption of joint convexity is satisfied thanks to a direct application of the Lyapounov Theorem. In Appendix B we show that there are other interesting non trivial cases in which joint convexity holds and in particular we focus our attention on the case of unanimity games. In fact given the tight link that there is between invariant biseparable preferences and the Choquet capacities [7] studying the cases in which joint convexity holds requires understanding up to which point Lyapounov Theorem can be extended to capacities.

Definition 15 *(GMM (2004) [15]) Let* $f, g \in \mathcal{F}$. *Then the unambiguous preference* $\succsim_i^*$ *w.r.t. to the completion criteria* $\succsim_i$ *is such that* $f \succsim_i^* g$ *if* $\lambda \cdot f + (1-\lambda) \cdot h \succsim_i^* \lambda \cdot g + (1-\lambda) \cdot h$ *for all* $\lambda \in (0,1]$ *and* $h \in \mathcal{F}$.

We now have all the elements that allow us to properly formalize Proposition 16.

Proposition 16 *The following statements are equivalent:*

1. $\succeq^*$ *is a Bewley preference;* $\{\succeq_i\}_{i=1}^N$ *and* C^o *are Invariant Biseparable and satisfy joint convexity;* $\{\succeq_i\}_{i=1}^N$ *satisfy Consistency Axiom 11 w.r.t.* $\succeq^*$*;* C^o *satisfies Consistency Toward Criteria Axiom 12 w.r.t.* $\{\succeq_i\}_{i=1}^N$*;*

2. *There exists a nonempty closed and convex set* $\mathcal{C}^*$ *of probabilities on* Σ*, a set of closed and convex sets* $\{\mathcal{C}^i\}_{i=1}^N \subseteq \mathcal{C}^*$ *of probabilities on* Σ*, a non constant affine function* $u : X \to \mathbb{R}$*, several monotonic, constant additive and positively homogenous linear functionals* $\{I_i : B_0(\Sigma) \to \mathbb{R}\}_{i=1}^N$ *and* $I_o : B_0(\Sigma) \to \mathbb{R}$ *and a set of non negative weights* $\{\gamma_i\}_{i=1}^N$ *such that the following holds:*

$$f \succeq^* g \;\Leftrightarrow \int_S E_{f(s)} u \cdot dp(s) \geq \int_S E_{g(s)} u \cdot dp(s) \;\; \forall p \in \mathcal{C}^* \tag{1}$$

$$f \succeq_i g \;\Leftrightarrow I_i(u(f)) \geq I_i(u(g)) \quad i = 1, \ldots, N \tag{2}$$

$$C^o(A) = \underset{f \in A}{argmax} \{I_o(u(f))\} = \underset{f \in A}{argmax} \left\{ \sum_{j=1}^N \gamma_j \cdot I_j(u(f)) \right\} \tag{3}$$

$$f \succsim_i^* g \iff \int_S E_{f(s)} u \cdot dp(s) \geq \int_S E_{g(s)} u \cdot dp(s) \;\; \forall p \in \mathcal{C}^i$$

Moreover in this case, $\mathcal{C}^*$ *and* $\{\mathcal{C}^i\}_{i=1}^N$ *are unique,* u *is unique up to positive affine transformations. If the set of outcomes* X *is finite we have that* $\sum_{j=1}^N \gamma_j = 1$.

[7] A preference relation is biseparable if it can be represented by a functional I on acts such that $I(x_A y) = \nu(A) u(x) + (1 - \nu(A)) u(y)$ for some utility function $u(\cdot)$ and some capacity $\nu(\cdot)$ where we assume that $\nu(\Omega) = 1$.

Proposition 16 shows that the Consistency Toward Criteria Axiom 12 simply implies that the final implemented choice correspondence is a linear combination with non negative weights of the functionals associated to the several completion criteria. As a consequence a DM that wants to be consistent w.r.t. the criteria that she considers possible in the first stage of the completion process will evaluate acts f by choosing a point of the convex cone spanned by the vector $I = (I_1(\cdot), \ldots, I_N(\cdot))$. This result is in line with the one obtained by Harsanyi (1955) [18] for social welfare functions by imposing a Pareto condition that plays the role of our consistency assumption. The key technical ideas behind this first basic Proposition are contained in De Meyer and Mongin (1995) [10]. From an historical perspective this result can be sawn as a generalization and an axiomatic foundation of the Hodges and Lehmann (1952) [19]'s paper that, for the first time in a statistical decision theoretic framework, provided a decision criteria that was a combination of the Wald's minimax principle and of the Bayesian approach.

4 A pessimistic attitude

In this section we will assume that we face a pessimistic DM that has a negative attitude toward the potential completion criteria. In particular, we formalize this pessimistic behavior by using an axiom that is the straightforward adaptation to our framework of the Expert Uncertainty Aversion axiom of Cres, Gilboa and Vieille (2011) [9] (CGV hereafter). In order to properly state the axiom we need to introduce some notation. For each act $f \in \mathcal{F}$ we denote by $c_i^f \in \mathcal{F}_c$ the certainty equivalent of the act f with respect to the preference relation $\succeq_i$. The certainty equivalent of act $f \in \mathcal{F}$ with respect to the choice correspondence $C^o : \Im \to \Im$ is defined as the constant act $c_o^f \in \mathcal{F}_c$ such that we have both $c_o^f \in C^o\left(\left\{f, c_o^f\right\}\right)$ and $f \in C^o\left(\left\{f, c_o^f\right\}\right)$.

Axiom 17 *(Criteria Uncertainty Aversion) For every $f \in \mathcal{F}$, $f_j \in \mathcal{F}$ $j = 1, \ldots J$, and every number $\alpha_j \geq 0$ such that $\sum_{j=1}^{J} \alpha_j = 1$, if $f \succeq_i \sum_{j=1}^{J} \alpha_j \cdot c_i^{f_j}$ for $i = 1, \ldots, N$ then $f \in C^o\left(\left\{f, \sum_{j=1}^{J} \alpha_j \cdot c_o^{f_j}\right\}\right)$.*

Notice that by simply setting $f_j = g$ and $\alpha_j = 1$ for any $j = 1, \ldots J$ we have that Criteria Uncertainty Aversion Axiom 17 implies Consistency Toward Criteria Axiom 12. Mathematically this latter axiom implies that the functional associated to the representation of the choice correspondence C^o is concave. The next Proposition 18 analitically formalize the attitude of our pessimistic DM.

Proposition 18 *The following statements are equivalent:*

1. *$\succeq^*$ is a Bewley preference; $\{\succeq_i\}_{i=1}^N$ and C^o are invariant biseparable; $\{\succeq_i\}_{i=1}^N$ satisfy Consistency Axiom 11 w.r.t. $\succeq^*$; C^o satisfies Criteria Uncertainty Aversion Axiom 17 w.r.t. $\{\succeq_i\}_{i=1}^N$;*

2. *There exists a nonempty closed and convex set $\mathcal{C}^*$ of probabilities on Σ, a set of closed and convex sets $\{\mathcal{C}^i\}_{i=1}^N \subseteq \mathcal{C}^*$ of probabilities on Σ, a non constant affine function $u: X \to \mathbb{R}$, several monotonic, constant additive and positively homogenous linear functionals $\{I_i : B_0(\Sigma) \to \mathbb{R}\}_{i=1}^N$ and $I_o : B_0(\Sigma) \to \mathbb{R}$ and a closed and convex set $\Gamma \subseteq \triangle(\{1,2,\ldots,N\})$ such that for every $f, g \in \mathcal{F}$*

$$f \succeq^* g \;\Leftrightarrow\; \int_S E_{f(s)}u \cdot dp(s) \geq \int_S E_{g(s)}u \cdot dp(s) \;\; \forall p \in \mathcal{C}^*$$

$$f \succeq_i g \;\Leftrightarrow I_i(u(f)) \geq I_i(u(g)) \quad i = 1, \ldots, N$$

$$C^o(A) = \underset{f \in A}{argmax}\ \{I_o(u(f))\} = \underset{f \in A}{argmax} \left\{ \min_{\gamma \in \Gamma} \sum_{j=1}^N \gamma_j \cdot I_j(u(f)) \right\}$$

$$f \succsim_i^* g \iff \int_S E_{f(s)}u \cdot dp(s) \geq \int_S E_{g(s)}u \cdot dp(s) \;\; \forall p \in \mathcal{C}^i$$

Moreover in this case, $\mathcal{C}^*$ and $\{\mathcal{C}^i\}_{i=1}^N$ are unique, u is unique up to positive affine transformations.

Proposition 18 generalizes and extends to our framework the main representation theorem of CGV [9]. Notice that CGV assume GS [17]'s preferences while we make the weaker assumption of Invariant Biseparable preference relations. The other results contained in Proposition 18 are standard and in line with what showed by GMMS [16] and GMM [15]. We will now introduce a corollary of Proposition 18 that clarifies under which conditions we end up in a representation theorem a là GMMS [16]. Before stating the corollary we need to recall the definition of Caution Axiom introduced by GMMS [16]. In particular two preference relations $\succeq^*$ and $\succeq_i$ satisfy Caution if the following holds:

Axiom 19 *(Caution) For $f \in \mathcal{F}$ and $g \in \mathcal{F}_c$, $f \not\succeq^* g$ implies $g \succeq_i f$.*

Corollary 20 *Under the assumptions of Proposition 18, if the there exists $\bar{i} \in \{1,\ldots,N\}$ such that $\succeq_{\bar{i}}$ satisfies Caution Axiom 19 with respect to $\succeq^*$ and the standard vector $e_{\bar{i}} \in \Gamma \subseteq \triangle(\{1,2,\ldots,N\})$, where $e_{\bar{i}}$ is the standard vector of $\mathbb{R}^N$ that assigns weight 1 to the element in position $\bar{i}$, then there exists a nonempty closed and convex set $\mathcal{C}^*$ of probabilities on Σ, a non constant affine function $u: X \to \mathbb{R}$ such that:*

$$f \succeq^* g \;\Leftrightarrow \int_S E_{f(s)}u \cdot dp(s) \geq \int_S E_{g(s)}u \cdot dp(s) \;\; \forall p \in \mathcal{C}^*$$

$$C^o(A) = \underset{f \in A}{argmax} \left\{ \min_{p \in \mathcal{C}^*} \int_S E_{f(s)}u \cdot dp(s) \right\}$$

Moreover, in this case, $\mathcal{C}^*$ is unique and u is unique up to positive affine transformations.

Corollary 20 gives us a representation identical to the one of GMMS [16]. Notice that we didn't assume that any of the subjective potential completion criteria satisfies uncertainty aversion and moreover we didn't assume that the subjective choice correspondence C^o, that is indeed implemented, satisfies either uncertainty aversion or cautiousness. The only assumption that we made on C^o is that it has a pessimistic attitude toward the potential completion criteria.

Hence Criteria Uncertainty Aversion Axiom 17 alone is not sufficient for having a representation result a là GMMS [16] even if $N-1$ of the potential criteria satisfy caution. It is in fact necessary that our DM considers possible to use only the cautious completion criteria in order to collapse in GMMS [16]'s main theorem and as a consequence only a really cautious agent will satisfy this type of representation theorem. We will now introduce a novel behavioral axiom that we will use in order to characterize the behavior of an agent a là GMMS [16]. One of the most pessimistic attitude that a DM can have toward the potential completion criteria is described by the following axiom:

Axiom 21 *(Caution Toward Criteria) For $f \in \mathcal{F}$ and $g \in \mathcal{F}_c$ if $\exists i \in \{1, 2, \ldots, N\}$ such that $f \not\succeq_i g$ then $g \in C^o(\{f, g\})$.*

Proposition 22 relaxes Criteria Uncertainty Aversion Axiom 17 and by assuming that the DM satisfies Consistency Toward Criteria Axiom 12 and Caution Toward Criteria Axiom 21 it axiomatically funds the behavior of a really cautious agent.

Proposition 22 *The following statements are equivalent:*

1. *If $\succeq^*$ is a Bewley preference; $\{\succeq_i\}_{i=1}^N$ and C^o are invariant biseparable; $\{\succeq_i\}_{i=1}^N$ satisfy Consistency Axiom 11 w.r.t. $\succeq^*$; C^o satisfies Uniformity Toward Criteria Axiom 12 and Caution Toward Criteria Axiom 21 w.r.t. $\{\succeq_i\}_{i=1}^N$; $\exists \bar{i} \in \{1, \ldots, N\}$ such that $\succeq_{\bar{i}}$ satisfies Caution Axiom 19 w.r.t. $\succeq^*$.*

2. *There exists a nonempty closed and convex set $\mathcal{C}^*$ of probabilities on Σ, a set of closed and convex sets $\{\mathcal{C}^i\}_{i=1}^N \subseteq \mathcal{C}^*$ of probabilities on Σ, a non constant affine function $u : X \to \mathbb{R}$, several monotonic, constant additive and positively homogenous linear functionals $\{I_i : B_0(\Sigma) \to \mathbb{R}\}_{i=1}^N$ and $I_o : B_0(\Sigma) \to \mathbb{R}$ and a set of non negative weights $\{\gamma_i\}_{i=1}^N$ such that:*

$$f \succeq^* g \;\Leftrightarrow \int_S E_{f(s)}u \cdot dp(s) \geq \int_S E_{g(s)}u \cdot dp(s) \;\; \forall p \in \mathcal{C}^*$$

$$f \succeq_i g \;\Leftrightarrow I_i(u(f)) \geq I_i(u(g)) \;\; i = 1, \ldots, N$$

$$C^o(A) = \underset{f\in A}{argmax}\, \{I_o(u(f))\} = \underset{f\in A}{argmax} \left\{ \sum_{j=1}^{N} \gamma_j \cdot I_j(u(f)) \right\} = \underset{f\in A}{argmax} \left\{ \min_{p\in\mathcal{C}^*} \int_S E_{f(s)}u \cdot dp(s) \right\}$$

$$f \succsim_i^* g \iff \int_S E_{f(s)}u \cdot dp(s) \geq \int_S E_{g(s)}u \cdot dp(s) \;\; \forall p \in \mathcal{C}^i$$

Moreover in this case, $\mathcal{C}^*$ and $\{\mathcal{C}^i\}_{i=1}^N$ are unique, u is unique up to positive affine transformations.

5 A variational approach

In this section we are going to axiomatically fund the behavior of a DM that is naturally biased toward some weights given to the potential completion criteria. In fact, even if the agent knows several potential completion criteria, he could find easier to implement only a specific subset of these criteria. This idea is mathematically formalized by giving to the functional representation associated to the choice correspondence C^o a variational representation a là MMR [20].

We introduce two axioms that are key for the representation contained in Theorem 25. The first axiom, called Criteria Betweeness, wants to capture the idea that the final valuation $I_o(u(f))$ of a generic non constant act f has to be always between the valuations of 2 potential completion criteria:

Axiom 23 *(Criteria Betweeness) For any $f \in \mathcal{F} \backslash \mathcal{F}_c$ there exist 2 indices $i,\, j \in \{1, 2, \ldots, N\}$ such that the following holds:*

$$f \in C^o\left(\left\{f,\, c_i^f\right\}\right)$$

and

$$c_j^f \in C^o\left(\left\{f,\, c_j^f\right\}\right)$$

While Volatility Axiom 24 formalizes the idea that the potential completion criteria considered need to have enough variability in their valuations in such a way that for any 2 non constant acts $f,\, g$ there always exists a criteria i such that $|I_i(u(f)) - I_i(u(g))| \geq |I_o(u(f)) - I_o(u(g))|$:

Axiom 24 *(Volatility) For any $f, g \in \mathcal{F} \setminus \mathcal{F}_c$ there exists an index $i \in \{1, 2, \ldots, N\}$ such that the following holds:*

$$\frac{1}{2} \cdot c_o^f + \frac{1}{2} \cdot c_i^g \in C^o\left(\left\{\frac{1}{2} \cdot c_o^f + \frac{1}{2} \cdot c_i^g,\, \frac{1}{2} \cdot c_o^g + \frac{1}{2} \cdot c_i^f\right\}\right)$$

We say that the choice correspondence C^o is Weak Variational [8] if it satisfies the Weak Axiom of Revealed Preferences Axiom 28, Non Triviality Axiom 29, Monotonicity Axiom 30, Continuity Axiom 31, Weak Certainty Independence Axiom 33.

Theorem 25 *Assume that $\{\succeq_i\}_{i=1}^N$ satisfy joint convexity. The following statements are equivalent:*

1. *$\succeq^*$ is a Bewley preference; $\{\succeq_i\}_{i=1}^N$ are Invariant Biseparable; $\{\succeq_i\}_{i=1}^N$ satisfy Consistency Axiom 11 w.r.t. $\succeq^*$; C^o is Weak Variational and it satisfies Criteria Uncertainty Aversion Axiom 17, Criteria Betweeness Axiom 23 and Volatility Axiom 24 w.r.t. $\{\succeq_i\}_{i=1}^N$.*

[8] In Appendix A it is possible to find the full statements of the axioms that characterize a weak variational choice correspondence and of Weak Certainty Independence Axiom 33.

2. *There exists a nonempty closed and convex set $\mathcal{C}^*$ of probabilities on Σ, , a set of closed and convex sets $\{\mathcal{C}^i\}_{i=1}^N \subseteq \mathcal{C}^*$ of probabilities on Σ, a non constant function $u : X \to \mathbb{R}$, several monotonic, constant additive and positively homogenous linear functionals $\{I_i : B_0(\Sigma) \to \mathbb{R}\}_{i=1}^N$; a normalized, monotonic, translation invariant functional $I_o : B_0(\Sigma) \to \mathbb{R}$ and a lower semicontinuous, grounded and convex function $c : \triangle(\{1, 2, \ldots, N\}) \to [0, \infty]$ such that for every $f, g \in \mathcal{F}$*

$$f \succeq^* g \ \Leftrightarrow \int_S E_{f(s)} u \cdot dp(s) \geq \int_S E_{g(s)} u \cdot dp(s) \ \ \forall p \in \mathcal{C}^*$$

$$f \succeq_i g \ \Leftrightarrow I_i(u(f)) \geq I_i(u(g)) \ \ i = 1, \ldots, N$$

$$C^o(A) = \underset{f \in A}{argmax} \ \{I_o(u(f))\} = \underset{f \in A}{argmax} \left\{ \min_{\gamma \in \triangle(\{1,2,\ldots,N\})} \left(\sum_{j=1}^N \gamma_j \cdot I_j(u(f)) + c(\gamma) \right) \right\}$$

$$f \succsim_i^* g \iff \int_S E_{f(s)} u \cdot dp(s) \geq \int_S E_{g(s)} u \cdot dp(s) \ \ \forall p \in \mathcal{C}^i$$

Moreover, in this case, $\mathcal{C}^*$ is unique and u is unique up to positive affine transformations.

For coherence with what we developed in the previous sections we decided to retain Criteria Uncertainty Aversion Axiom 17 in the statement of Theorem 25, even though it could be enough to have a weakened version of Criteria Uncertainty Aversion Axiom 17 of the following type:

Axiom 26 *(Weak Criteria Uncertainty Aversion) For every act $f, g, h \in \mathcal{F}$ and every number $\alpha \in (0, 1)$ if $f \succeq_i \alpha \cdot c_i^g + (1-\alpha) \cdot c_i^h$ for $i = 1, \ldots, N$ then $f \in C^o\left(\left\{f, \alpha \cdot c_o^g + (1-\alpha) \cdot c_o^h\right\}\right)$.*

As usual Proposition 18 can be viewed as almost [9] a special case of Theorem 25 if we have that $c(\gamma) = 0$ when $\gamma \in \Gamma$ and $c(\gamma) = +\infty$ when $\gamma \notin \Gamma$.

6 Related Literature

The papers that inspired the present work are mainly related either to the decision theoretic literature or to the social choice literature (in particular the problem of social aggregation). Our paper, like those of CGV [9] and of Nascimento (2012) [21], lies at the intersection of these two streams of literature. In this section we will briefly sum up the papers that are tightly related with our work and we will try to highlight the connections.

As mentioned in the Introduction section, our paper can be viewed as a generalization of GMMS [16]. In particular GMMS [16], by using a framework that involves one incomplete "objective" preference relation $\succeq^*$ and one complete "subjective" preference relation $\succeq\hat{}$, provide a new axiomatic foundation of the maxmin functional representation à la GS

[9] Notice that in Proposition 18 we don't have the assumption of joint convexity.

[17] without using the uncertainty aversion axiom. In order to link the "objective" and the "subjective" preference relations they make use of a standard Consistency Axiom and for the first time in the decision theoretic literature they introduce the Caution Axiom. This latter axiom states that the Decision Maker, when comparing a constant act with a generic non constant act, prefers the constant act whenever the choice is undecidable from an "objective" point of view. Their main theorem, from a functional representation point of view, is equivalent to our Proposition 22.

Cerreia-Vioglio (2012) [4] generalizes the results of GMMS [16] by weakening the Caution axiom and by using the Risk Independence axiom instead of the Constant Independence axiom. In this way the author is able to give an axiomatic foundation to effectively bounded Uncertainty averse preferences. The cost that the author has to pay in order to have very general results is to restrict attention to the class of preferences that satisfy the Unboundness axiom [10].

CGV [9] consider the problem of a DM that has to aggregate several experts opinions in order to make a decision in a framework characterized by uncertainty. Given that the experts have to provide an opinion to a unique individual, all of them use the same utility function when evaluating the different acts and by assumption all the experts have GS [17]'s maxmin preferences. Notice that in our work the fact that all preference relations have the same utility function is not imposed but it is a result of the construction of the model. Moreover the potential completion criteria that we study, i.e. invariant biseparable preferences, are generalizations of the GS [17]'s maxmin preferences. CGV [9] introduce a really interesting axiom, whose name is Expert Uncertainty Aversion, that axiomatically formalize a pessimistic behavior toward experts' opinions. This axiom delivers the concavity of the functional that it is necessary in order to make a separation argument à la GS. Our Axiom 17 is an adaptation to potential completion criteria of CGV [9]'s Expert Uncertainty Aversion.

Nascimento (2012) [21] studies the problem of aggregating preference orderings under "subjective" uncertainty. The author considers an "ex ante" preference $\succeq^{ex-ante}$ defined over set of lotteries of acts $\triangle(\mathcal{F})$, that represents the choices of the DM, and a set P of admissible "ex-post" preferences, that represents all experts' opinions. Like CGV [9], he needs to assume, by using a Weak Agreement Axiom, that the admissible class of preferences that he considers agrees over the set of constant acts. Some of the Propositions proposed by Nascimento [21] are highly in line with the Propositions and Theorem presented in our paper. In particular our Proposition 16, Proposition 18 and Theorem 25 can be viewed as a version of Theorem 1, Proposition 8 and Theorem 2 respectively of Nascimento (2012) [21] under the assumption that $\succeq^{ex-ante}$ satisfies reduction of compound

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Axiom 27 *(Unboundedness) A preference relation $\succeq$ satisfies unboundness if for each $x,\ y \in \mathcal{F}_c$ with $x \succ y$ then $\exists z,\ w \in \mathcal{F}_c$ such that the following holds:*

$$\frac{1}{2}z + \frac{1}{2}y \succeq x \succ y \succeq \frac{1}{2}w + \frac{1}{2}x$$

lotteries. Nascimento (2012) [21] has also a version of the main representation theorem of GMMS [16] that goes under the name of Proposition 7. Although there are some similarities between our work and the one by Nascimento (2012) [21], our paper is characterized by a totally different axiomatic structure. Moreover even if Nascimento(2012) [21] allows for infinite dimensional class of opinions while our work is completely based on a finite dimensional setup, his model needs to use an outer layer of randomization, given that $\succeq^{ex-ante}$ is defined over set of lotteries of acts $\triangle(\mathcal{F})$ in a Seo(2009) [24] style. On the other side our work is based on axioms defined over the set of Anscombe and Aumann acts $\mathcal{F}$ and Nascimento(2012) [21] himself in the "Final Remarks" section claims to be "... mainly interested in the restriction of the ex ante preference to the set of acts..." and not in the lotteries over acts. Finally, notice that our job was inspired by the problem of integrating an "objective" information inside a "subjective" framework while Nascimento(2012) [21]'s model doesn't fit well this type of issue as he explains again in the "Final Remarks" paragraph.

We want to conclude this section by briefly describing two papers that, like our paper, try to insert an "objective" datum inside the "subjective" framework of the DM.

Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2013a) [7] build up a model that merges a "subjective" choice framework a là Savage with a waldean "objective" piece of information. In fact, even though the work of Savage was prevalently inspired by the Waldean decision theoretic approach, Savage had a purely subjective setup and he didn't consider the classical "objective" datum of the problem. The authors, by using a consistency axiom, are able from one side to reveal the fact that the DM is indeed aware of the datum of the problem (see the structurally rich representation of their Proposition 1 for instance) and from the other side they are able to discriminate among models (see their Proposition 2). As a consequence the classical Savage criterion can be simply considered as the information that an outside observer, that doesn't know the "objective" datum, could collect from the agent's behavior. The structural representations proposed by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2013a) [7] in Proposition 1, Proposition 3 and Proposition 4 are particularly insightful because they make clear the distinction between state uncertainty and model uncertainty. This latter type of uncertainty, even if it is not directly payoff relevant, plays an instrumental role relative to state uncertainty.

Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2013b) [6] propose a framework in which they try to join the decision theoretic literature on ambiguity and the robust statistics literature on prior uncertainty. In particular they study the conditions under which a problem of ambiguity can be rephrased and reinterpreted in terms of prior uncertainty. The main tools that the authors use in order to join ambiguity and prior uncertainty are a Consistency Axiom and the concept of Dynkin space. Their Consistency Axiom, that is a version of the GMMS [16]'s Consistency Axiom, allow to make choices coherent with the probabilistic information by linking the family of "objective" rational beliefs P with the DM's "subjective" preference relation $\succeq$. The Dynkin space is the mathematical tool that Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2013b) [6] use in order to formally model the lack of information that affects the economic agent and the relevant probabilistic information is formalized as a sub $\sigma-$algebra of the $\sigma-$algebra of events. The authors interpret the set of strong extreme points of P $S(P)$

as the pure models of the problem and the DM knows that if the missing information was available the correct model would almost surely belong to the set $S(P)$. Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2013b) [6] have several representation results and in Theorem 4 they also have a variational representation a là MMR [20].

7 Appendix A

In this section of the Appendix we describe the axioms that we use in order to characterize our choice correspondence $C^o : \Im \to \Im$.

Axiom 28 *(Weak Axiom of Revealed Preferences) If $A, B \in \Im$ are such that $B \subseteq A$ and $C^o(A) \cap B \neq \emptyset$ then $C^o(B) = C^o(A) \cap B$;*

Axiom 29 *(Nontriviality) There are $f, g \in \mathcal{F}$ such that $f = C^o(\{f, g\})$;*

Axiom 30 *(Monotonicity) For every $f, g \in \mathcal{F}$, if $f(s) \in C^o(\{f(s), g(s)\})$ for all $s \in S$ implies $f \in C^o(\{f, g\})$;*

Axiom 31 *(Continuity) For any $f, g, h \in \mathcal{F}$ we have that the sets:*

$$\{\lambda \in [0, 1] : \lambda \cdot f + (1-\lambda) \cdot g \in C^o(\{\lambda \cdot f + (1-\lambda) \cdot g, h\})\}$$

and

$$\{\lambda \in [0, 1] : h \in C^o(\{\lambda \cdot f + (1-\lambda) \cdot g, h\})\}$$

are closed in $[0, 1]$ *;*

Axiom 32 *(Constant Independence) For every $A \in \Im$, $h \in \mathcal{F}_c$ and $\alpha \in (0, 1)$ $C^o(\alpha \cdot A + (1-\alpha) \cdot h) = \alpha \cdot C^o(A) + (1-\alpha) \cdot h$;*

Axiom 33 *(Weak Certainty Independence) For every $A \in \Im$, $h, z \in \mathcal{F}_c$ and $\alpha \in (0, 1]$ if $C^o(\alpha \cdot A + (1-\alpha) \cdot h) = \alpha \cdot C^o(A) + (1-\alpha) \cdot h$ then $C^o(\alpha \cdot A + (1-\alpha) \cdot z) = \alpha \cdot C^o(A) + (1-\alpha) \cdot z$;*

Axiom 34 *(Risk Independence) For every $A \in \Im_c$, $h \in \mathcal{F}_c$ and $\alpha \in (0, 1)$ we have that $C^o(\alpha \cdot A + (1-\alpha) \cdot h) = \alpha \cdot C^o(A) + (1-\alpha) \cdot h$ where $\Im_c$ is the set of all non empty finite subsets of $\mathcal{F}_c$.*

We also introduce two axioms for a generic preference relation $\succsim$ that will be used later in the proofs of the different Propositions.

Axiom 35 *(Risk Independence)For every $f, g, h \in \mathcal{F}_c$ and $\alpha \in (0, 1]$ $f \succ g$ if and only if $\alpha \cdot f + (1-\alpha) \cdot h \succ \alpha \cdot g + (1-\alpha) \cdot h$;*

Axiom 36 *(Weak Certainty Independence) For every $f, g \in \mathcal{F}$ $h, z \in \mathcal{F}_c$ and $\alpha \in (0, 1)$ if $\alpha \cdot f + (1-\alpha) \cdot h \succeq \alpha \cdot g + (1-\alpha) \cdot h$ then $\alpha \cdot f + (1-\alpha) \cdot z \succeq \alpha \cdot g + (1-\alpha) \cdot z$.*

8 Appendix B

In this part of the appendix we will show that there are interesting non trivial cases in which the assumption of Joint Convexity holds by focusing our attention on the case of unanimity games. In particular we will try to approach this research question by understanding under which conditions we can extend the Lyapounov Theorem when we deal with unanimity games, i.e. the simplest types of capacities we can think of.

8.1 Simple results for unanimity games

Fix $A \in \Sigma$ and define the unanimity game $\nu_A : \Sigma \to \mathbb{R}$ as:

$$\nu_A(B) = \begin{cases} 1 & if\ B \subseteq A \\ 0 & else \end{cases}$$

The choquet integral w.r.t. the unanimity capacity is given by $\int f \cdot d\nu_A = \inf_{\omega \in A} f(\omega)$.

Definition 37 *We say that a set* $\{\nu_{A_i}\}_{i=1}^n$ *of unanimity games have sufficient disjoint support if* $\forall i \in \{1, 2, \ldots, n\}$ *we have that* A_i *is not a subset of* $\bigcup_{j \neq i} A_j$.

Proposition 38 *Let* $\{\nu_{A_i}\}_{i=1}^n$ *be a set of unanimity games with sufficient disjoint support. Then the set* I *defined as follows:*

$$I = \left\{ \int u(f) \cdot d\nu_{A_1}, \ldots, \int u(f) \cdot d\nu_{A_n} | \ u(f) \ \ is\ \Sigma - measurable\ and\ 0 \leq u(f) \leq 1 \right\}$$

is a convex subset of $\mathbb{R}^n$.

Proof. We will show by construction that the set obtained under the assumptions of the proposition is the hypercube of length 1 in $\mathbb{R}^n$. Consider the following function:

$$u(f) = \begin{cases} \alpha_1 & if\ A_1 \backslash \bigcup_{j \neq 1} A_j \\ \alpha_2 & if\ A_2 \backslash \bigcup_{j \neq 2} A_j \\ & \vdots \\ \alpha_n & if\ A_n \backslash \bigcup_{j \neq n} A_j \\ 1 & otherwise \end{cases}$$

and notice that $\forall i \in \{1, 2, \ldots, n\}$ we have that $\int u(f) \cdot d\nu_{A_i} = \min_{\omega \in A} u(f(\omega)) = \alpha_i$ and by letting vary each α_i in the interval $[0,\ 1]$ we obtains each point of the hypercube. ■

Example 39 *As a simple example we show that, differently w.r.t. the general case of Proposition 38, for the special case of* 2 *unanimity games we don't need the assumption of sufficient disjoint support*[11].

Proposition 40 *Let* ν_{A_1} *and* ν_{A2} *be two unanimity games. Then the set* I *defined as follows:*

$$I = \left\{ \int u(f) \cdot d\nu_{A_1}, \int u(f) \cdot d\nu_{A_2} | \ u(f) \ \textit{is } \Sigma - \textit{measurable and } 0 \leq u(f) \leq 1 \right\}$$

is a convex subset of $\mathbb{R}^2$.

Proof. Notice that in general we can have 2 cases: either the sufficient support assumption holds and we know that the result is true by Proposition 38 or the sufficient support assumption doesn't hold. Hence the strategy will be to assume that the sufficient support assumption doesn't hold and we will show that I is still a convex subset of $\mathbb{R}^2$. Suppose w.l.o.g. that $A_1 \subseteq A_2$. Assume that we want to plot I on a plane where we have on the axes $I_1(u(f)) = \int u(f) \cdot d\nu_{A_1}$ and $I_2(u(f)) = \int u(f) \cdot d\nu_{A_2}$. If $A_1 = A_2$ we get that I is the segment joining the points $(0,0)$ and $(1,1)$ on the line $I_1(u(f)) = I_2(u(f))$ and of course it is convex. If $A_1 \subset A_2$, then in this case we have that $I_2(u(f)) \leq I_1(u(f))$ and I is equal to a square triangle that is of course a convex set. ■

Example 41 *After seeing the result contained in Example 39 we could think that the assumption of disjoint supports is indeed too strong and that we can relax it still obtaining that the set* I *is a convex subset of* $\mathbb{R}^n$. *Indeed the next Proposition provide a negative answer to this latter intuition.*

Proposition 42 *Let* $\{\nu_{A_i}\}_{i=1}^n$ *be a set of unanimity games. Then the set* I *defined as follows:*

$$I = \left\{ \int u(f) \cdot d\nu_{A_1}, \ldots, \int u(f) \cdot d\nu_{A_n} | \ u(f) \ \textit{is } \Sigma - \textit{measurable and } 0 \leq u(f) \leq 1 \right\}$$

is not in general a convex subset of $\mathbb{R}^n$.

Proof. The strategy of the proof is to construct a counterexample in $\mathbb{R}^3$ in which I is not a convex set. Fix $n = 3$ and consider the case in which $A_1 \subset (A_2 \cup A_3)$, $A_2 \cap A_3 = \emptyset$, $A_1 \cap A_3 \neq \emptyset$, $A_1 \cap A_2 \neq \emptyset$. Notice that the points $(0,\ 1,\ 0)$, $(0,\ 0,\ 1) \in I$. To see that the latter statement is true it is enough to consider the following functions:

[11] We thank Simone Cerreia Vioglio who focused our attention on this particular case by showing us two interesting non trivial examples.

$$u\left(f^{(0,\ 1,\ 0)}\right)=\begin{cases}1 & if\ A_1\cap A_2\\ 1 & if\ A_2\backslash A_1\\ 0 & if\ A_1\cap A_3\\ 0 & if\ A_3\backslash A_1\\ 1 & otherwise\end{cases}$$

$$u\left(f^{(0,\ 0,\ 1)}\right)=\begin{cases}0 & if\ A_1\cap A_2\\ 0 & if\ A_2\backslash A_1\\ 1 & if\ A_1\cap A_3\\ 1 & if\ A_3\backslash A_1\\ 1 & otherwise\end{cases}$$

On the other side if we combine with weight $\frac{1}{2}$ the points $(0,\ 1,\ 0)$ and $(0,\ 0,\ 1)$ we obtain the point $\left(0, \frac{1}{2},\ \frac{1}{2}\right) \notin I$ because from how we constructed the capacities the first component of the vector has to be always bigger than the minimum of the other two components of the vector. ■

9 Appendix C

Lemma 43 . *Define the relation $\succeq^o$ as follows:*

$$f \succeq^o g \Leftrightarrow f \in C^o\left(\{f,\ g\}\right)$$

The roles of $\succsim^o$ and of C^o are perfectly interchangeable

Proof. By Weak Axiom of Revealed Preferences 28, $\succeq^o$ is a preference relation that satisfies Completeness Axiom 9, Transitivity Axiom 3 and Reflexivity Axiom 2. If the choice correspondence satisfies c-Independence Axiom 32, Weak Certainty Independence Axiom 33, Risk Independence Axiom 34, Continuity Axiom 31 and Monotonicity Axiom 30 we have that $\succeq^o$ satisfies c-Independence Axiom 10, Weak Certainty Independence Axiom 36, Risk Independence Axiom 35, Continuity Axiom 6 and Monotonicity Axiom 5. Moreover $\succeq^o$ has to satisfy Uncertainty Toward Criteria Axiom 17, Uniformity Toward Criteria Axiom 12, Cautiousness Toward Criteria Axiom 21 if the choice correspondence satisfies the corresponding axioms. ■

Given that by Lemma 43 $\succeq^o$ perfectly represents C^o all the proofs will be given in terms of $\succeq^o$.

Proof. of Proposition 16.

We prove only the sufficiency part given that the necessity part is standard.

The core part of the proof is an adaptation to our framework of the proof of Proposition 1 of De Meyer and Mongin (1995) [10]. For the standard representations of the Knightian preference relation $\succeq^*$ and of the invariant biseparable preferences check the proof of Proposition 18. Let's define $I^{full} = \left(I_o\left(\cdot\right), I_1\left(\cdot\right), \ldots, I_N\left(\cdot\right)\right)$ and notice that the range of I^{full} is convex because $\{\succeq_i\}_{i=1}^N$ and C^o satisfy joint convexity. Let's define $M =$

$I^{full} - I^{full} = \left\{x - y \mid x, y \in I^{full}\right\}$ and $Z = \{z_0 < 0, z_1 \geq 0, \ldots, z_N \geq 0\}$. Notice that M is nonempty, convex, symmetric w.r.t. the origin and it contains the origin. Moreover the affine hull of M, denoted as $aff(M)$, by Theorem 1.1 of Rockafellar (1970) [22] coincides with the vector subspace spanned by M. The fact that C^o satisfies consistency toward criteria implies that $aff(M) \cap Z = \emptyset$. To see this assume by way of contradiction that $\exists z \in aff(M) \cap Z$ and given that $z \in aff(M)$ we have that $z = \sum_{j=1}^{N} \lambda_j \cdot z_j$ with $z_j \in M \;\; for \; j = 1, \ldots, N$. The fact that M is symmetric w.r.t. the origin allows us to assume that $\lambda_j \geq 0 \, for \, j = 1, \ldots, N$, in fact $-z_j \in M$ if $z_j \in M$. Now notice that by the convexity of the set M we have that $\frac{z}{\sum_{j=1}^{N} \lambda_j} = \sum_{j=1}^{N} \frac{\lambda_j}{\sum_{j=1}^{N} \lambda_j} \cdot z_j \in M \cap Z$ and we reached a contradiction with the fact that C^o satisfies consistency toward criteria. Let's now notice that $\left(\overline{Z} - e_o\right) \subset Z$, where $\overline{Z}$ denotes the closure of the set Z and e_o is the standard vector of R^{N+1} that assigns value 1 to the first component. Hence we have that $aff(M) \cap \left(\overline{Z} - e_o\right) = \emptyset$ and both these sets are polyhedral non empty convex sets. As a consequence we can strictly separate $aff(M)$ and $\left(\overline{Z} - e_o\right)$ and we get that $\exists \mu \in R^{N+1}$ such that $\mu \cdot (x - e_o) > \mu \cdot y$ for any $x \in \overline{Z}$ and $y \in aff(M)$. Given that $0 \in aff(M)$ we have that $\mu_o < 0$ (notice that $0 \in \overline{Z}$). Moreover the fact that $aff(M)$ is a vector subspace implies that $\mu \cdot y = 0$ for any $y \in aff(M)$. Hence it is possible to fix $g \in F$ and we have that for any $f \in F$ the following holds:

$$I_o(f) = \sum_{j=1}^{N} \left(\frac{\mu_j}{-\mu_o}\right) I_j(f) + \sum_{j=1}^{N} \left(\frac{\mu_j}{-\mu_o}\right) (-I_j(g)) - I_o(g) = \sum_{j=1}^{N} \gamma_j \cdot I_j(f) + kost$$

where $\gamma_j = \frac{\mu_j}{-\mu_o}$ for any $j = 1, \ldots, N$, $kost = \sum_{j=1}^{N} \left(\frac{\mu_j}{-\mu_o}\right) (-I_j(g)) - I_o(g)$ and the fact that by Lemma 1 of GMM [15] $I_o(f)$ is unique up to positive affine transformations allow us to assume w.l.o.g. that $kost = 0$.

If we assume that X is finite, by c-completeness of $\succeq^*$ and by the fact that $\{\succeq_i\}_{i=1}^{N}$ are consistent with respect to $\succeq^*$ we can assume w.l.o.g. that $\exists x^*, x_* \in X$ such that $I_j(x^*) = 1$ and $I_j(x_*) = -1$ for any $j = 1, \ldots, N$ and the result $\sum_{j=1}^{N} \gamma_j = 1$ follows. ■

Proof. of Proposition 18.

We want to show that if

$$C^o(A) = \underset{f \in A}{argmax} \{I_o(u(f))\} = \underset{f \in A}{argmax} \left\{ \underset{\gamma \in \Gamma}{min} \sum_{i=1}^{N} \gamma_i \cdot I_I(u(f)) \right\}$$

then C^o satisfies Criteria Uncertainty Aversion Axiom 19. Fix $f \in \mathcal{F}$, $f_j \in \mathcal{F}$ and $\alpha_j \geq 0$ for $j = 1, \ldots J$ such that $\sum_{j=1}^{J} \alpha_j = 1$ and $f \succeq_i \sum_{j=1}^{J} \alpha_j \cdot c_i^{f_j}$ for $i = 1, \ldots, N$. Set $A = \left\{ f; \; \sum_{j=1}^{J} \alpha_j \cdot c_i^{f_j} \right\}$ and by Lemma 43 it is enough to show the result for $\succeq^o$. Notice that if $f \succeq_i \sum_{j=1}^{J} \alpha_j \cdot c_i^{f_j}$ then $I_i(u(f)) \geq \sum_{j=1}^{J} \alpha_j \cdot I_i(u(f_j))$ hence we have that for any set of weights $\gamma_i \geq 0$ for $i = 1, \ldots N$ such that $\sum_{i=1}^{N} \gamma_i = 1$ the following holds:

$$\sum_{i=1}^{N}\gamma_i \cdot I_i\left(u\left(f\right)\right) \geq \sum_{i=1}^{N}\gamma_i \cdot \left(\sum_{j=1}^{J}\alpha_j \cdot I_i\left(u\left(f_j\right)\right)\right) = \sum_{j=1}^{J}\alpha_j \cdot \left(\sum_{i=1}^{N}\gamma_i \cdot I_i\left(u\left(f_j\right)\right)\right)$$

. By taking the *min* of both sides over the set of weights Γ and by using the properties of the min function we obtain the following chain of inequalities:

$$\begin{aligned} \min_{\gamma\in\Gamma}\sum_{i=1}^{N}\gamma_i \cdot I_i\left(u\left(f\right)\right) = I_o\left(u\left(f\right)\right) &\geq \min_{\gamma\in\Gamma}\sum_{j=1}^{J}\alpha_j \cdot \left(\sum_{i=1}^{N}\gamma_i \cdot I_i\left(u\left(f_j\right)\right)\right) \\ &\geq \sum_{j=1}^{J}\alpha_j \cdot \min_{\gamma\in\Gamma}\left(\sum_{i=1}^{N}\gamma_i \cdot I_i\left(u\left(f_j\right)\right)\right) = \sum_{j=1}^{J}\alpha_j \cdot I_o\left(u\left(f_j\right)\right) \end{aligned}$$

.

The rest of the necessity part is standard and it is left to the reader.

We now prove the sufficiency part given.

By Theorem 1 of GMMS [16] there exists u^* and $\mathcal{C}^*$ such that:

$$f \succeq^* g \;\; \Leftrightarrow \int_S E_{f(s)}u^* \cdot dp\left(s\right) \geq \int_S E_{g(s)}u^* \cdot dp\left(s\right) \;\; \forall p \in \mathcal{C}^*$$

Set $f \succeq_i' g$ if and only if $\lambda \cdot f + (1-\lambda) \cdot h \succeq_i \lambda \cdot g + (1-\lambda) \cdot h$ for all $\lambda \in [0,1]$ and $h \in \mathcal{F}$. By Lemma 1, Proposition 5 and Proposition 7 of GMM [15] we know that for each $i = o, 1, \ldots, N$ there exists a non empty closed and convex set $\mathcal{C}^i$ of probabilities on Σ, a non constant function $u^i : X \to \mathbb{R}$ and a monotonic, constant additive and positively homogenous linear functional $I_i : B_0\left(\Sigma\right) \to \mathbb{R}$ (where with $B_0\left(\Sigma\right)$ we denote the vector space generated by the indicator functions of the elements of Σ endowed with the supnorm metric) such that for every $f, g \in \mathcal{F}$ we have that:

$$f \succeq_i' g \;\; \Leftrightarrow \int_S E_{f(s)}u^i \cdot dp\left(s\right) \geq \int_S E_{g(s)}u^i \cdot dp\left(s\right) \;\; \forall p \in \mathcal{C}^{\mathrm{i}}$$

$$f \succeq_i g \;\; \Leftrightarrow I_i\left(u^i\left(f\right)\right) \geq I_i\left(u^i\left(g\right)\right)$$

$$\min_{p\in\mathcal{C}^i}\int_S E_{f(s)}u^i \cdot dp\left(s\right) \leq I_i\left(u^i\left(f\right)\right) \leq \max_{p\in\mathcal{C}^i}\int_S E_{f(s)}u^i \cdot dp\left(s\right)$$

These 3 relations can be obtained for all $\{\succeq_i\}_{i=1}^N$ and $\succeq^o$.

Remember that Criteria Uncertainty Aversion Axiom 17 implies Consistency Toward Criteria Axiom 12 and that all the relations $\{\succeq_i\}_{i=1}^N$ are consistent with $\succeq^*$ and as a consequence also $\succeq^o$ is consistent with $\succeq^*$. $\succeq^*$ is a non trivial subrelation of $\{\succeq_i\}_{i=1}^N$ and $\succeq^o$on constant acts $\mathcal{F}_c$, by Proposition 4 of GMM [15] we can assume that $u^* = u^o = u^1 = \cdots = u^N := u$. Without loss of generality we will assume that 0 is in the interior of the range of the utility function u.

Again by Proposition 4 of GMM [15] we have that each one of $\succeq_i'$ is the maximal subrelation satisfying independence of the corresponding potential criteria $\succeq_i$. By consistency we have that $\succeq^* \subseteq \succeq_i$ and Proposition A.1 of GMM [15] delivers $\mathcal{C}^i \subseteq \mathcal{C}^*$. Hence we have that for any $f \in \mathcal{F}$:

$$\min_{p\in\mathcal{C}^*} \int_S E_{f(s)} u \cdot dp(s) \leq \min_{p\in\mathcal{C}^i} \int_S E_{f(s)} u \cdot dp(s) \leq I_i(u(f))$$

We denote with $R = R(I)$ the range of the vector $I = (I_1(\cdot), \ldots, I_N(\cdot))$ and it is easy to find a function $\phi : R \to \mathbb{R}$ such that for each $f \in \mathcal{F}$ we have that $I_o(f) = \phi(I(u(f)))$. By easily adapting the arguments contained in the proof of Theorem 1 of CGV [9], we can proceed by several extension of the functional ϕ up to $\mathbb{R}^N$ by preserving monotonicity, positive homogeneity, constant additivity and concavity[12]. For any $x \in \mathbb{R}^N$, let's denote with $\bar{\Psi}(x)$ the final extension ϕ up to $\mathbb{R}^N$.

Finally by a standard argument using the supporting hyperplane theorem, i.e. as in Lemma 3.5 of GS [17], for any fixed $x^* \in \mathbb{R}^N$ we can find $h^{x^*} \in \mathbb{R}^N$ and $\gamma^{x^*} \in \mathbb{R}$ such that the following holds:

$$h^{x^*} \cdot x + \gamma^{x^*} \geq \bar{\Psi}(x) \;\; for\, any\, x \in \mathbb{R}^N$$

$$h^{x^*} \cdot x^* + \gamma^{x^*} = \bar{\Psi}(x^*)$$

Notice that by positive homogeneity of the function $\bar{\Psi}(\cdot)$ we have that for any $\alpha > 0$ it is true that $\alpha \cdot h^{x^*} \cdot x^* + \gamma^{x^*} = \alpha \cdot \bar{\Psi}(x^*) \Rightarrow \gamma^{x^*} = 0$. Monotonicity of $\bar{\Psi}(\cdot)$ implies that $h^{x^*} \geq 0$, because if by contradiction we assume that $h_i^{x^*} < 0$ then we would have a violation of monotonicity, i.e. $\bar{\Psi}(x^*) = h^{x^*} \cdot x^* + \gamma^{x^*} > h^{x^*} \cdot (x^* + e_i) + \gamma^{x^*} \geq \bar{\Psi}(x^* + e_i)$. Finally if $\bar{\Psi}(x^*) = c$ we have that for any $\rho \in \mathbb{R}$ the following relationship holds $\bar{\Psi}(\rho \cdot x^* + (1-\rho) \cdot \vec{c}) = \bar{\Psi}(\rho \cdot x^*) + (1-\rho) \cdot c = \rho \cdot h^{x^*} \cdot x^* + (1-\rho) \cdot c = \rho \cdot \bar{\Psi}(x^*) + (1-\rho) \cdot c = c$ because $\bar{\Psi}(\cdot)$ satisfies constant additivity. This latter consideration in turn it implies that $h^{x^*} \cdot \vec{c} = c$ and it allow us to interpret h^{x^*} as a vector of positive weights that sum up to 1. Finally by setting $\Gamma := closure\left(co\left\{h^{x^*} \mid x^* \in \mathbb{R}^N\right\}\right)$, we have shown that there exists a closed and convex set $\Gamma \subseteq \triangle(\{1, 2, \ldots, N\})$ such that for all $x \in \mathbb{R}^N$:

$$\bar{\Psi}(x) = \min_{\gamma\in\Gamma} \sum_{j=1}^{N} \gamma_j \cdot x_j$$

and in particular we have that $\phi(x) = \min_{\gamma\in\Gamma} \sum_{j=1}^{N} \gamma_j \cdot x_j$ for all $x \in R$. ■

Proof. of Corollary 20. If there exists $\bar{\imath}$ such that $\succeq_{\bar{\imath}}$ satisfies Caution Axiom 19 then by Theorem 3 of GMMS [16] we have that:

$$\min_{p\in\mathcal{C}^*} \int_S E_{f(s)} u \cdot dp(s) = \min_{p\in\mathcal{C}^{\bar{\imath}}} \int_S E_{f(s)} u \cdot dp(s) = I_{\bar{\imath}}(u(f))$$

Given that for all $i = 1, \ldots, N$ we have $\min_{p\in\mathcal{C}^{\bar{\imath}}} \int_S E_{f(s)} u \cdot dp(s) = I_{\bar{\imath}}(u(f)) = \min_{p\in\mathcal{C}^*} \int_S E_{f(s)} u \cdot$

[12] The details are available from the authors upon request.

$dp(s) \leq \min_{p\in\mathcal{C}^i} \int_S E_{f(s)}u \cdot dp(s) \leq I_i(u(f))$ and that $e_{\overline{\imath}} \in \Gamma \subseteq \triangle(\{1,2,\ldots,N\})$ we have that for each $f \in \mathcal{F}$ the following holds:

$$I_o(f) = \min_{\gamma\in\Gamma} \sum_{j=1}^{N} \gamma_j \cdot I_j(u(f)) = \min_{p\in\mathcal{C}^*} \int_S E_{f(s)}u \cdot dp(s)$$

■

Proof. of Proposition 22.

The necessity part is standard and it is proved with the same arguments used in the previous propositions.

For the sufficiency argument we prove only the part that contains new ideas and that was not shown previously. Given that $\succeq^o$ satisfies Uniformity Toward Criteria Axiom 12 and the fact that $\{\succeq_i\}_{i=1}^N$ satisfy Consistency Axiom 11 with respect to $\succeq^*$ implies that $\succeq^o$ is consistent with respect to $\succeq^*$. Following the same type of arguments showed in Lemma 18 it is possible to show that there exists $\mathcal{C}^*$ and a unique u such that for any act $f \in \mathcal{F}$:

$$I_o(u(f)) \geq \min_{p\in\mathcal{C}^*} \int_S E_{f(s)}u \cdot dp(s)$$

If there exists $\overline{\imath}$ such that $\succeq_{\overline{\imath}}$ satisfies caution then by Theorem 3 of GMMS [16] we have that:

$$I_o(u(f)) \geq \min_{p\in\mathcal{C}^*} \int_S E_{f(s)}u \cdot dp(s) = \min_{p\in\mathcal{C}^{\overline{\imath}}} \int_S E_{f(s)}u \cdot dp(s) = I_{\overline{\imath}}(u(f))$$

Suppose by contra that $I_o(u(f)) > \min_{p\in\mathcal{C}^*} \int_S E_{f(s)}u \cdot dp(s)$, then it is possible to find a constant act $g \in \mathcal{F}_c$ such that the following holds:

$$I_o(u(f)) > u(g) > \min_{p\in\mathcal{C}^*} \int_S E_{f(s)}u \cdot dp(s) = I_{\overline{\imath}}(u(f))$$

But this latter inequality contradicts Caution Toward Criteria Axiom 21 because for $\overline{\imath}$ we have that $f \not\succeq_{\overline{\imath}} g$ but $f \succeq^o g$. ■

Proof. of Theorem 25.

Let' start with the necessity part. Given the functional representation of I_o we will show that the axiom betweness is satisfied. Otherwise by way of contradiction assume that:

$$I_o(u(f)) = \min_{\gamma\in\triangle(\{1,2,\ldots,N\})} \left(\sum_{j=1}^{N} \gamma_j \cdot I_j(u(f)) + c(\gamma) \right) > \max\ I_i(u(f))$$

and assume that the minimum is attained at $\overline{\gamma}$. Hence for any $\gamma \in \triangle(\{1,2,\ldots,N\})$ we have that:

$$\sum_{j=1}^{N} \gamma_j \cdot I_j(u(f)) + c(\gamma) \geq \sum_{j=1}^{N} \overline{\gamma}_j \cdot I_j(u(f)) + c(\overline{\gamma}) > \max_{i\in\{1,2,\ldots,N\}} I_i(u(f))$$

. Clearly $c(\overline{\gamma}) > 0$ and, given that c is grounded, $\exists \gamma^g \in \triangle(\{1,2,\ldots,N\})$ such that $c(\gamma^g) = 0$. These considerations lead to the following chain of inequalities:

$$\sum_{j=1}^{N} \gamma_j^g \cdot I_j(u(f)) \geq \sum_{j=1}^{N} \overline{\gamma}_j \cdot I_j(u(f)) + c(\overline{\gamma}) > \max_{i\in\{1,2,\ldots,N\}} I_i(u(f))$$

and we have that:

$$\sum_{j=1}^{N} \gamma_j^g \cdot I_j(u(f)) > \max_{i\in\{1,2,\ldots,N\}} I_i(u(f))$$

that is absurd. Let's verify the Volatility Axiom 24. By way of contradiction suppose that there exist $f, g \in \mathcal{F} \backslash \mathcal{F}_c$ such that either

$$\begin{aligned} &\min_{\gamma\in\triangle(\{1,2,\ldots,N\})} \left(\sum_{j=1}^{N} \gamma_j \cdot I_j(u(f)) + c(\gamma) \right) - \min_{\gamma\in\triangle(\{1,2,\ldots,N\})} \left(\sum_{j=1}^{N} \gamma_j \cdot I_j(u(g)) + c(\gamma) \right) \\ &< \min_{i\in\{1,2,\ldots,N\}} (I_i(u(f)) - I_i(u(g))) \end{aligned}$$

or

$$\begin{aligned} &\max_{i\in\{1,2,\ldots,N\}} (I_i(u(f)) - I_i(u(g))) \\ &< \min_{\gamma\in\triangle(\{1,2,\ldots,N\})} \left(\sum_{j=1}^{N} \gamma_j \cdot I_j(u(f)) + c(\gamma) \right) - \min_{\gamma\in\triangle(\{1,2,\ldots,N\})} \left(\sum_{j=1}^{N} \gamma_j \cdot I_j(u(g)) + c(\gamma) \right) \end{aligned}$$

. W.l.o.g. suppose that:

$$\min_{\gamma\in\triangle(\{1,2,\ldots,N\})} \left(\sum_{j=1}^{N} \gamma_j \cdot I_j(u(f)) + c(\gamma) \right) = \sum_{j=1}^{N} \overline{\gamma}_j \cdot I_j(u(f)) + c(\overline{\gamma})$$

and

$$\min_{\gamma\in\triangle(\{1,2,\ldots,N\})} \left(\sum_{j=1}^{N} \gamma_j \cdot I_j(u(g)) + c(\gamma) \right) = \sum_{j=1}^{N} \widetilde{\gamma}_j \cdot I_j(u(g)) + c(\widetilde{\gamma})$$

If

$$\begin{aligned} &\min_{\gamma\in\triangle(\{1,2,\ldots,N\})} \left(\sum_{j=1}^{N} \gamma_j \cdot I_j(u(f)) + c(\gamma) \right) - \min_{\gamma\in\triangle(\{1,2,\ldots,N\})} \left(\sum_{j=1}^{N} \gamma_j \cdot I_j(u(g)) + c(\gamma) \right) \\ &< \min_{i\in\{1,2,\ldots,N\}} (I_i(u(f)) - I_i(u(g))) \end{aligned}$$

then the contradiction is reached by the following chain of inequalities:

$$\begin{aligned} \min_{i\in\{1,2,\ldots,N\}} (I_i(u(f)) - I_i(u(g))) &\leq \sum_{j=1}^{N} \overline{\gamma}_j \cdot I_j(u(f)) + c(\overline{\gamma}) - \sum_{j=1}^{N} \overline{\gamma}_j \cdot I_j(u(g)) - c(\overline{\gamma}) \\ &\leq \sum_{j=1}^{N} \overline{\gamma}_j \cdot I_j(u(f)) + c(\overline{\gamma}) - \min_{\gamma\in\triangle(\{1,2,\ldots,N\})} \left(\sum_{j=1}^{N} \gamma_j \cdot I_j(u(g)) + c(\gamma) \right) \\ &< \min_{i\in\{1,2,\ldots,N\}} (I_i(u(f)) - I_i(u(g))) \end{aligned}$$

. On the other side if

$$\max_{i\in\{1,2,...,N\}} \left(I_i\left(u\left(f\right)\right) - I_i\left(u\left(g\right)\right)\right)$$
$$< \underset{\gamma\in\triangle(\{1,2,...,N\})}{min} \left(\sum_{j=1}^{N}\gamma_j \cdot I_j\left(u\left(f\right)\right) + c\left(\gamma\right)\right) - \underset{\gamma\in\triangle(\{1,2,...,N\})}{min} \left(\sum_{j=1}^{N}\gamma_j \cdot I_j\left(u\left(g\right)\right) + c\left(\gamma\right)\right)$$

then we can reach a contradiction as follows

$$\max_{i\in\{1,2,...,N\}} \left(I_i\left(u\left(f\right)\right) - I_i\left(u\left(g\right)\right)\right)$$
$$< \underset{\gamma\in\triangle(\{1,2,...,N\})}{min} \left(\sum_{j=1}^{N}\gamma_j \cdot I_j\left(u\left(f\right)\right) + c\left(\gamma\right)\right) - \sum_{j=1}^{N}\widetilde{\gamma}_j \cdot I_j\left(u\left(g\right)\right) - c\left(\widetilde{\gamma}\right)$$
$$\leq \sum_{j=1}^{N}\widetilde{\gamma}_j \cdot I_j\left(u\left(f\right)\right) + c\left(\widetilde{\gamma}\right) - \sum_{j=1}^{N}\widetilde{\gamma}_j \cdot I_j\left(u\left(g\right)\right) - c\left(\widetilde{\gamma}\right) \leq \max_{i\in\{1,2,...,N\}} \left(I_i\left(u\left(f\right)\right) - I_i\left(u\left(g\right)\right)\right)$$

.

Let's finally verify Criteria Uncertainty Aversion Axiom 17. Fix $f \in \mathcal{F}$, $f_j \in \mathcal{F}$ and $\alpha_j \geq 0$ for $j = 1,\dots J$ such that $\sum_{j=1}^{J}\alpha_j = 1$ and $f \succeq_i \sum_{j=1}^{J}\alpha_j \cdot c_i^{f_j}$ for $i = 1,\dots,N$. Set $A = \left\{f;\ \sum_{j=1}^{J}\alpha_j \cdot c_i^{f_j}\right\}$ and by Lemma 43 it is enough to show the result for $\succeq^o$. Notice that if $f \succeq_i \sum_{j=1}^{J}\alpha_j \cdot c_i^{f_j}$ then $I_i\left(u\left(f\right)\right) \geq \sum_{j=1}^{J}\alpha_j \cdot I_i\left(u\left(f_j\right)\right)$ hence we have that for any set of weights $\gamma_i \geq 0$ for $i = 1,\dots N$ such that $\sum_{i=1}^{N}\gamma_i = 1$ and any lower semicontinuous, grounded and convex function $c : \triangle\left(\{1,2,\dots,N\}\right) \to [0,\ \infty]$ the following holds:

$$\sum_{i=1}^{N}\gamma_i \cdot I_i\left(u\left(f\right)\right) + c\left(\gamma\right) \geq \sum_{i=1}^{N}\gamma_i \cdot \left(\sum_{j=1}^{J}\alpha_j \cdot I_i\left(u\left(f_j\right)\right)\right) + \sum_{j=1}^{J}\alpha_j \cdot c\left(\gamma\right) = \sum_{j=1}^{J}\alpha_j \cdot \left(\sum_{i=1}^{N}\gamma_i \cdot I_i\left(u\left(f_j\right)\right) + c\left(\gamma\right)\right)$$

. By taking the *min* of both sides over the set of weights Γ and by using the properties of the min function we obtain the following chain of inequalities:

$$\underset{\gamma\in\Gamma}{min}\left(\sum_{i=1}^{N}\gamma_i \cdot I_i\left(u\left(f\right)\right) + c\left(\gamma\right)\right) = I_o\left(u\left(f\right)\right) \geq \underset{\gamma\in\Gamma}{min}\sum_{j=1}^{J}\alpha_j \cdot \left(\sum_{i=1}^{N}\gamma_i \cdot I_i\left(u\left(f_j\right)\right) + c\left(\gamma\right)\right)$$
$$\geq \sum_{j=1}^{J}\alpha_j \cdot \underset{\gamma\in\Gamma}{min}\left(\sum_{i=1}^{N}\gamma_i \cdot I_i\left(u\left(f_j\right)\right) + c\left(\gamma\right)\right) = \sum_{j=1}^{J}\alpha_j \cdot I_o\left(u\left(f_j\right)\right)$$

.

The rest of the necessity arguments are standard and are left to the reader.

Let's now show the sufficiency part. As usual Criteria Uncertainty Aversion Axiom 17 implies Consistency Toward Criteria Axiom 12. For the standard representations of the Knightian preference relation $\succeq^*$ and of the Invariant Biseparable preferences check the proof of Proposition 18. We will focus our attention on the representation of the choice correspondence. By the Lemma 43 the revealed preference relation $\succeq^o$ associated to the choice correspondence C^o satisfies Completeness Axiom 9, Transitivity Axiom 3, Nontriviality Axiom 4, Monotonicity Axiom 5, Continuity Axiom 6, Weak Certainty Independence Axiom 36. Notice that Weak Certainty Independence Axiom 36 implies Risk Independence Axiom 35. The existence of a normalized, monotonic, continuous functional $I_o : B_0(\Sigma) \to \mathbb{R}$, given that our continuity axiom implies the archimedean continuity axiom, is the result of Proposition 1 of Cerreia Vioglio, Ghirardato, Maccheroni, Marinacci, Siniscalchi (2011) [5]. By the proof of Proposition 18 we know that there exists a unique utility function $u : X \to \mathbb{R}$ for all the functional representations of the preference relations $\succeq^*$, $\{\succeq_i\}_{i=1}^N$ and $\succeq^o$. We denote with $R = R(I)$ the range of the vector $I = (I_1(\cdot), \dots, I_N(\cdot))$ and given that $\succeq^o$ satisfies Consistency Toward Criteria Axiom 12 we can find a well defined function $\phi : R \to \mathbb{R}$ such that for each $f \in \mathcal{F}$ we have that $I_o(u(f)) = \phi(I(u(f)))$. Moreover $\phi(\cdot)$ is monotone: fix 2 acts $f, g \in \mathcal{F}$ such that $I(u(f)) \geq I(u(g))$ and by Consistency Toward Criteria Axiom 12 we have that $\phi(I(u(f))) = I_o(u(f)) \geq I_o(u(g)) = \phi(I(u(g)))$. $\phi(\cdot)$ is also normalized, given that for any $k \in \mathbb{R}$ such that $k \cdot \mathbf{1}_N \in R$ we have that there exists a constant act $c^k \in \mathcal{F}_c$ such that for all $i \in \{o, 1, \dots, N\}$ $I_i\left(u\left(c^k\right)\right) = u\left(c^k\right) = k$. It is important to notice that by assumption the domain of ϕ is convex subset of $B_o(\Gamma)$, where $\Gamma = 2^N$, given that $\{\succeq_i\}_{i=1}^N$ satisfy joint convexity. Now we want to show that ϕ is a niveloid, i.e. for any $x, y \in R$ we have that $\phi(x) - \phi(y) \leq \underset{i\in\{1,2,\dots,N\}}{max} (x_i - y_i)$. If $f^x, f^y \in \mathcal{F}_c$ then the relation trivially holds by Consistency Toward Criteria Axiom 12:

$$\phi(x) - \phi(y) = I_o(u(f^x)) - I_o(u(f^y)) = \underset{i\in\{1,2,\dots,N\}}{max} (x_i - y_i) =$$
$$\underset{i\in\{1,2,\dots,N\}}{max} (I_i(u(f^x)) - I_i(u(f^y))) = u(f^x) - u(f^y)$$

If $f^x, f^y \in \mathcal{F} \setminus \mathcal{F}_c$ then by Volatility Axiom 24 we have that $\exists i \in \{1, 2, \dots, N\}$ such that the following holds:

$$\frac{1}{2} \cdot c_o^{f^y} + \frac{1}{2} \cdot c_i^{f^x} \succeq^o \frac{1}{2} \cdot c_o^{f^x} + \frac{1}{2} \cdot c_i^{f^y}$$

and we have that $u\left(\frac{1}{2} \cdot c_o^{f^y} + \frac{1}{2} \cdot c_i^{f^x}\right) \geq u\left(\frac{1}{2} \cdot c_o^{f^x} + \frac{1}{2} \cdot c_i^{f^y}\right)$. Notice that $I_o\left(u\left(c_i^{f^x}\right)\right) = \sum_{x\in X} c_i^{f^x}(x) \cdot u(x) = I_i\left(u\left(c_i^{f^x}\right)\right)$ and by using the affinity of the utility function we have that:

$$\frac{1}{2} \cdot u\left(c_o^{f^y}\right) + \frac{1}{2} \cdot u\left(c_i^{f^x}\right) \geq \frac{1}{2} \cdot u\left(c_o^{f^x}\right) + \frac{1}{2} \cdot u\left(c_i^{f^y}\right)$$

that it is equivalent to the following:

$$\frac{1}{2} \cdot I_o\left(u\left(c_o^{f^y}\right)\right) + \frac{1}{2} \cdot I_i\left(u\left(c_i^{f^x}\right)\right) \geq \frac{1}{2} \cdot I_o\left(u\left(c_o^{f^x}\right)\right) + \frac{1}{2} \cdot I_i\left(u\left(c_i^{f^y}\right)\right)$$

and as a consequence we have that:

$$\frac{1}{2} \cdot I_o(u(f^y)) + \frac{1}{2} \cdot I_i(u(f^x)) \geq \frac{1}{2} \cdot I_o(u(f^x)) + \frac{1}{2} \cdot I_i(u(f^y))$$

By simplifying and rearranging the last expression we obtain:

$$\phi(x) - \phi(y) = I_o(u(f^x)) - I_o(u(f^y)) \leq I_i(u(f^x)) - I_i(u(f^y))$$

$$\leq \underset{i\in\{1,2,\ldots,N\}}{max} \left(I_i\left(u\left(f^x\right)\right) - I_i\left(u\left(f^y\right)\right)\right) = \underset{i\in\{1,2,\ldots,N\}}{max} \left(x_i - y_i\right)$$

Finally if $f^x \in \mathcal{F} \setminus \mathcal{F}_c$ and $f^y \in \mathcal{F}_c$ by Criteria Betweenness Axiom 23 we have that $\exists i \in \{1, 2, \ldots, N\}$ such that $c_i^{f^x} \succeq^o f^x$, i.e. $I_o\left(u\left(f^x\right)\right) \leq I_o\left(u\left(c_i^{f^x}\right)\right) = u\left(c_i^{f^x}\right) = I_i\left(u\left(f^x\right)\right)$. Notice that by assumption $I_o\left(u\left(f^y\right)\right) = u\left(f^y\right) = I_i\left(u\left(f^y\right)\right)$ and we have that $I_o\left(u\left(f^x\right)\right) + u\left(f^y\right) - u\left(f^y\right) \leq I_i\left(u\left(f^x\right)\right)$. The last expression implies that the following is true:

$$\phi(x) - \phi(y) = I_o\left(u\left(f^x\right)\right) - I_o\left(u\left(f^y\right)\right) \leq I_i\left(u\left(f^x\right)\right) - I_i\left(u\left(f^y\right)\right)$$
$$\leq \underset{i\in\{1,2,\ldots,N\}}{max} \left(I_i\left(u\left(f^x\right)\right) - I_i\left(u\left(f^y\right)\right)\right) = \underset{i\in\{1,2,\ldots,N\}}{max} \left(x_i - y_i\right)$$

Let's now verify that $\phi(\cdot)$ is concave. For any $x,\, y \in R$ and $\alpha \in (0,\, 1)$ we want to show that $\phi(\alpha \cdot x + (1-\alpha) \cdot y) \geq \alpha \cdot \phi(x) + (1-\alpha) \cdot \phi(y)$. By the convexity of R we know that there exists $f^{xy} \in \mathcal{F}$ such that $I\left(u\left(f^{xy}\right)\right) = \alpha \cdot x + (1-\alpha) \cdot y$. For any $i \in \{1, 2, \ldots, N\}$ we have that $I_i\left(u\left(f^{xy}\right)\right) = \alpha \cdot I_i\left(u\left(f^x\right)\right) + (1-\alpha) \cdot I_i\left(u\left(f^y\right)\right) = \alpha \cdot I_i\left(u\left(c_i^{f^x}\right)\right) + (1-\alpha) \cdot I_i\left(u\left(c_i^{f^y}\right)\right)$, i.e. $f^{xy} \sim^i \alpha \cdot c_i^{f^x} + (1-\alpha) \cdot c_i^{f^y}$. By Criteria Uncertainty Aversion Axiom 17 we have that $f^{xy} \succeq^o \alpha \cdot c_o^{f^x} + (1-\alpha) \cdot c_o^{f^y}$, i.e. (by using the affinity of u) $I_o\left(u\left(f^{xy}\right)\right) \geq \alpha \cdot I_o\left(u\left(f^x\right)\right) + (1-\alpha) \cdot I_o\left(u\left(f^y\right)\right)$ and as a consequence the following is true:

$$\phi(\alpha \cdot x + (1-\alpha) \cdot y) = \phi\left(I\left(u\left(f^{xy}\right)\right)\right) \geq \alpha \cdot \phi\left(I\left(u\left(f^x\right)\right)\right) + (1-\alpha) \cdot \phi\left(I\left(u\left(f^y\right)\right)\right) = \alpha \cdot \phi(x) + (1-\alpha) \cdot \phi(y)$$

Hence $\phi(\cdot)$ is a concave niveloid defined over a convex subset of $B(\Gamma)$ and by following Dolecki and Greco (1995) [11] we know that the least niveloid on $B_o(\Gamma)$ that extends $\phi(\cdot)$ for any $x \in B_o(\Gamma)$ is:

$$\hat{\phi}(x) = \underset{y\in R}{sup} \left[\phi(y) + \underset{i\in\{1,\ldots,\, N\}}{inf} \left(x_i - y_i\right)\right]$$

By following the same approach of MMR [20], we can apply the Fenchel-Moreau theorem to $\hat{\phi}(\cdot)$ and we have that the following functional representation holds:

$$\hat{\phi}(x) = \underset{\mu\in ba(\Gamma)}{min} \left[\langle x,\, \mu\rangle - \hat{\phi}^*(\mu)\right]$$

where $ba(\Gamma)$ denote the set of bounded and finitely additive set functions $\mu : \Gamma \to \mathbb{R}$ endowed with the total variation norm and $\hat{\phi}^*(\mu) = \underset{x\in B(\Gamma)}{inf} \left(\langle x,\, \mu\rangle - \hat{\phi}(x)\right)$ is the Fenchel conjugate of $\hat{\phi}(\cdot)$. If μ is not positive then, given that w.l.o.g. we can assume that $0 \in int(u(X))$ and $[-1,\, 1] \subseteq u(X)$, it is possible to find $x \geq 0$ such that $\langle x,\, \mu\rangle < 0$. As a consequence for all $\alpha \geq 0$ we have that the monotonicity of $\hat{\phi}(\cdot)$ implies that $\langle \alpha \cdot x,\, \mu\rangle - \hat{\phi}(\alpha \cdot x) \leq \alpha \cdot \langle x,\, \mu\rangle - \hat{\phi}(0)$ and $\hat{\phi}^*(\mu) = -\infty$. If $\mu(\{1, 2, \ldots, N\}) \neq 1$ then for any $x \in B(\Gamma)$ and $a \in \mathbb{R}$ by using the constant additivity of the niveloid we have that the following holds:

$$\langle x+a,\, \mu\rangle - \hat{\phi}\,(x+a) = \langle x,\, \mu\rangle - \hat{\phi}\,(x) + a\cdot(\mu\,(\{1,2,\ldots,N\}) - 1)$$

and again $\hat{\phi}^*\,(\mu) = -\infty$. As a consequence of the previous argument we have the following representation:

$$\hat{\phi}\,(x) = \min_{\gamma\in\triangle(\Gamma)} \left[\langle x,\, \gamma\rangle - \hat{\phi}^*\,(\gamma)\right] = \min_{\gamma\in\triangle(\{1,2,\ldots,N\})} \left(\sum_{j=1}^{N} \gamma_j \cdot I_j\,(u\,(f^x)) + c\,(\gamma)\right)$$

where $c\,(\gamma) = -\hat{\phi}^*\,(\gamma)$. The fact that $c\,(\cdot)$ is a lower semicontinuous, grounded and convex function is a direct application of Lemma 26 of MMR [20]. ■

References

[1] F. Anscombe and R. Aumann, A definition of subjective probability, *The Annals of Mathematical Statistics*, 34, 1, 199-205, 1963.

[2] R. Aumann, Utility theory without the completeness axiom, *Econometrica*, 30,445–462, 1962.

[3] T. Bewley, Knightian decision theory. Part I, *Decisions in Economics and Finance*, 25, 2, 79-110, 2002.

[4] S. Cerreia-Vioglio, Objective Rationality and Uncertainty Averse Preferences, *mimeo*, 2012.

[5] S. Cerreia-Vioglio, P. Ghirardato, F. Maccheroni, M. Marinacci and M. Siniscalchi, Rational preferences under ambiguity, *Economic Theory*, 48, 2-3, 341-375, 2011.

[6] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci and L. Montrucchio, Ambiguity and robust statistics, *Journal of Economic Theory*, 148, 3, 974-1049, 2013b.

[7] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci and L. Montrucchio, Classical subjective expected utility, *Proceedings of the National Academy of Sciences*, 110,17, 6754-6759, 2013a.

[8] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci and A. Rustichini, Niveloids and Their Extensions: Risk Measures on Small Domains, *Igier Working Paper, 458,* 2012.

[9] H. Cres, I. Gilboa and N. Vieille, Aggregation of multiple prior opinions, *Journal of Economic Theory*, 146, 6, 2563-2582, 2011.

[10] B. De Meyer and P. Mongin, A note on affine aggregation, *Economics Letters*, 47, 2, 177–183, 1995.

[11] S. Dolecki and G. H. Greco, Niveloids, *Topological Methods in Nonlinear Analysis*, 5, 1–22, 1995.

[12] J. Dubra, F. Maccheroni and E. Ok, Expected utility theory without the completeness axiom, Journal of Economic Theory, 115, 1, 118-133, 2004.

[13] D. Ellsberg, Risk, ambiguity, and the Savage axioms, *The Quarterly Journal of Economics*, 643-669, 1961.

[14] P. Fishburn, Utility theory for decision making, *New York: Wiley*, 1970.

[15] P. Ghirardato, F. Maccheroni and M. Marinacci, Differentiating ambiguity and ambiguity attitude, *Journal of Economic Theory*, 118, 2, 133-173, 2004.

[16] I. Gilboa, F. Maccheroni, M. Marinacci and D. Schmeidler, Objective and subjective rationality in a multiple prior model, *Econometrica*, 78, 2, 755-770, 2010.

[17] I. Gilboa and D. Schmeidler, Maxmin expected utility with non-unique prior, *Journal of Mathematical Economics*, 18, 2, 141-153, 1989.

[18] J. C. Harsanyi, Cardinal welfare, individual ethics, and interpersonal comparisons of utility, *The Journal of Political Economy*, 309-321, 1955.

[19] J. Hodges and E. Lehmann, The use of previous experience in reaching statistical decisions, *The Annals of Mathematical Statistics*, 23, 3, 396-407,1952.

[20] F. Maccheroni, M. Marinacci and A. Rustichini, Ambiguity aversion, robustness, and the variational representation of preferences, *Econometrica*, 74, 6, 1447-1498, 2006.

[21] L. Nascimento, The ex ante aggregation of opinions under uncertainty, *Theoretical Economics*, 7, 3, 535–570, 2012.

[22] R. Rockafellar, Convex Analysis, *Princeton University Press*, 1970.

[23] D. Schmeidler, Subjective probability and expected utility without additivity, *Econometrica*, 57, 3, 571–587, 1989.

[24] K. Seo, Ambiguity and Second-Order Belief, *Econometrica*, 77, 5, 1575–1605, 2009.

[25] L.S. Shapley and M. Baucells Multiperson Utility, *Working Paper*, N. 779, Department of Economics, UCLA, 1998.