International transmission of bubble crashes

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Preliminary Version

Abstract

We are interested in the impact and the transmission of a bubble burst arising in a country to another country. We extend the overlapping generations model with stochastic bubbles developed by Weil (1987) in which the burst of the bubble is based on self-fulfilling beliefs (sunspot) to an open economy with two large identical countries, information symmetry and homogeneous beliefs. In each country, a bubble represented by an asset with no fundamental value is supplied and traded across countries. We show that even though the realization of sunspots is specific to one country, the sunspot is spread to other country through the portfolio choice of agents. When only the foreign country is subject to sunspots, the burst of the foreign bubble generates a change in confidence in the domestic bubble involving its immediate collapse. When both countries are subject to sunspots, agents take into account the risk of the domestic bubble collapse in their decisions. Therefore, the burst of the foreign bubble can generate asymmetric effects on the domestic one. The domestic bubble can persist at a higher value just after the burst of Foreign bubble, but may deflate over time.

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1 Introduction

History of financial markets overflows with examples of financial crises starting with the the burst of Dutch tulip bubble during 17th. The crash of a speculative bubble are often at the root of financial crises. As illustrated during the subprime crisis, financial crises are contagious phenomena. Started in the U.S, it propagated to other countries. From Kindleberger (1978), we could consider the burst of Sea South bubble and Mississippi Bubble occurring in 1720 an example of contagion.[1] Because of globalization, private institutions hold international diversified portfolio making financial interdependent, and promoting thus the transmission of financial crises across countries. Our paper focus on transmission of bubble burst across asset markets in a simple dynamic general equilibrium model.

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[1] For further details about bubbles, financial crisis and contagion in history, the reader can refer to Kindleberger and Aliber (2005).

Some contributions on financial crises and contagion provide an explanation based on information asymmetry (among others, Calvo and Mendoza (2000) and Kodres and Pritsker (2002)) (To be developped?). In our paper, agents' expectations and beliefs are only the key determinants underlying international transmission of financial crises across countries.

We extend the overlapping-generations exchange economy with stochastic bubbles developed by Weil (1987) in which bubbles have a constant probability to burst into a two-country model with information symmetry and homogeneous beliefs.[2] The two countries are perfectly symmetric in all respects. There is a unique perishable good, and a bubble represented by an asset with no fundamental value is supplied by each country and traded across borders. Following Weil (1987), all agents believe that the the foreign (domestic) bubble bursts if a sunspot occur in the foreign (domestic) country.

We derive conditions for both bubbles to occur and analyze the response of the domestic bubble to the burst of the foreign bubble. When only the foreign country is subject to sunspots, a stationary equilibrium with domestic and foreign bubbles exists before the burst of the foreign bubble provided that the foreign bubble is not too risky. We find the similar condition as in Weil (1987), namely a sufficiently high probability for the foreign bubble to persist. In such an economy, the burst of the foreign bubble generates a change in confidence in the domestic bubble involving its immediate collapse. Even though the realization of sunspots is specific to one country, the financial crisis arising in the foreign country propagates to other country through the portfolio of agents. International portfolio diversification makes asset markets interdependent. The risk of foreign bubble to crash is thus spread to the domestic bubble.

When both countries are subject to sunspots, the scope for the existence of a stationary equilibrium with both bubbles before the bursting of the foreign bubble is reduced if agents expect that the value of the domestic bubble is zero just after the burst of the foreign bubble. Only sufficiently safe bubbles can exist. This equilibrium is likely to occur in a dynamically inefficient economy only if the probability for Foreign and Home bubbles to persist is closed to 1. Furthermore, we show that the burst of the foreign bubble does not necessarily cause the immediate crash of the domestic bubble. Since agents internalize the risk of the domestic bubble to explode, the domestic bubble can persist even at a higher value just after the burst of the foreign bubble, then deflate over time.

Our paper is closely related to the literature on rational bubbles (Samuelson (1958), Tirole (1985), Weil (1987), and Santos and Woodford (1997)). Since Tirole's seminal papers (1982, 1985), the overlapping generation model is a useful framework to study rational bubble. However, Tirole (1985) studies rational bubbles which never burst. To tackle this problem, Weil (1987) extends Tirole's framework by introducing a probability for the bubble to burst. Referring to sunspot theory, agents coordinate their beliefs in the bubble with respect to the realization of sunspots which occurs with a non-zero probability. In his paper, he shows that stochastic bubbles are likely to occur if agents have sufficiently confidence in their existence.[3] Building on Weil (1987), our paper shows that only safe bubble can exist in an open economy.

Recent papers analyze the effect of bubble and its burst on the real economic activity (Kocherlakota (2009), Farhi and Tirole (2010), Hirano and Yanagawa (2010), Martin and Ventura (2012), Miao and Wang (2012)). Introducing financial frictions, they show that the burst of the bubble generates a recession with a drop of investment and output. Furthermore, these analyzes are done in a closed economy. In contrast to these contributions, we consider only the financial sphere of

[2] We abstract for the production side of the economy to highlight contagion effects arising in asset markets.

[3] Wigniolle (2014) extends the conditions for the existence of stochastic bubbles in an overlapping generations model by considering rank-dependent utility function.

the economy studying the international transmission of a bubble crash arising in one country to another bubble.

To the best of our knowledge, few paper has jointly analyzed the existence of stochastic bubbles in open economy and the international transmission of bubble crashes. Tandon and Wang (2003) analyze bubbles*à la* Weil in a small open economy. In their paper, only small open economy is affected by sunspots. Therefore, the burst of the bubble has no impact on the bubble issued by the rest of the world. In our paper, we reverse this result by considering a two-country model. Considering a world economy of several countries, Ventura (2008) studies the international transmission of a bubble crash between countries with heterogeneous fundamentals. He shows that a bubble crash arising in an developing country affects emergent and developed economies. In our paper, we argue that an international transmission of bubble crash can occur between countries which are perfectly identical in terms of fundamentals.

This paper is organized as follows. In the next section, we present the closed economy model and analyze the dynamic properties. Section 3 extends the model to a two-country model. In section 4, we analyze the transmission of Foreign bubble burst to Home bubble when Home country is immune against sunspot. Section 5 studies a world economy when neither Home country nor Foreign country is immune against sunspot. Concluding remarks are provided in Section 6, and all the proofs are gathered in a final Appendix.

2 A Closed Economy Framework

The world economy consists of two large countries $i = \{H, F\}$ perfectly identical with respect to preferences, endowments and population size. H refers to Home country, and F to Foreign country. Each country is characterized by a closed overlapping generations exchange economy *à la* Weil (1987).

For simplicity, we assume no population growth, and at each date t, a generation of unit size is born in each country. There is a single perishable good, and each agent is endowed with y_1 units of this good when young and y_2 when old. In their first period of life, they consume an amount c_t^i of the good, and d_{t+1}^i in their second period of life.

Furthermore, agents can save through intrinsically useless asset papers supplied by their country in a constant amount normalized to one. Since these assets have no fundamental value, a positive price $Q_t^i > 0$ depicts a bubble. The value of the bubble Q_t^i follows a random process depending on the realization of an extrinsic random variable (a sunspot) in country i. There are two states of nature: sunspot and no sunspot. A sunspot occurs with probability $1 - \pi \in [0, 1)$, and no sunspot with probability π. Suppose that the economy of country i is in a state of nature with no sunspots in period t, all agents living in country i expects that the bubble in country i persists ($Q_{t+1}^i > 0$) in period $t + 1$ if no sunspot occurs, and that the bubble bursts ($Q_{t+1}^i = 0$) if a sunspot occurs. After the burst, the bubble cannot reappear, and the economy of country i behaves as an economy without bubble.

Strictly speaking, we suppose that the bubble Q_t^i follows a Markov process with the following transition matrix:

As underlined by Weil (1987), π can be seen as a measure of the confidence in the bubble Q_t^i. In our paper, it is exogenous and constant through time. For instance, the burst of Foreign bubble arising in Foreign country does not modify the confidence of agents living in Home country in the value of their bubble Q_t^i.

	$Q_t^i = t > 0$	$Q_t^i = 0$
$Q_{t+1}^i = q_{t+1}^i > 0$	π	0
$Q_{t+1}^i = 0$	$1-\pi$	1

The representative agent of a generation born at period t in country i faces the following problem:

$$\max_{c_t, d_{t+1} \geq 0} \quad \mathbb{E}_t \left[u(c_t^i) + v(d_{t+1}^i) \right] \tag{1}$$

$$s.t. \quad c_t^i + Q_t^i x_t^i = y_1 \tag{2}$$

$$d_{t+1}^i = y_2 + Q_{t+1}^i x_t^i \tag{3}$$

where

y_1 = endowment in the first period of life

y_2 = endowment in the second period of life

c_t^i = consumption by the agent in her first period of life

d_{t+1}^i = consumption by the agent in her second period of life

x_t^i = amount of asset held by the agent

Assumption 1 *$u\left(c^i\right)$ and $v\left(d^i\right)$ are continuous functions defined on $[0, +\infty)$, C^2 on $(0, +\infty)$, strictly increasing $(u' > 0, v' > 0)$ and concave $(u'' < 0, v'' < 0)$. Moreover, we assume that $\lim_{c \to 0} u'\left(c^i\right) = +\infty$, $\lim_{d^i \to 0} v'\left(d^i\right) = +\infty$, and $\varepsilon_v(d^i) \equiv -dv''(d)/v'(d) \leq 1$.*

When none of the bubble is not valued in period t, $Q_t^i = 0$, the economy of country i is in the autarky regime. There are no intergenerational trades.

When the bubble in country i is valued in period t, $Q_t^i = q_t^i > 0$. The optimal behavior of the agent living in country i is summarized by the following intertemporal trade-off:

$$\frac{u'(y_1 - q_t^i x_t^i)}{v'(y_2 + q_{t+1}^i x_t^i)} = \pi \frac{q_{t+1}^i}{q_t^i} \tag{4}$$

Asset market clearing conditions imply $x_t^i = 1$. We assume that the economy initially starts with a bubble, and study equilibrium dynamics up to the random date on which a sunspot occur in country i involving the burst of the bubble Q_t^i.[4] An equilibrium is a sequence $\left\{q_t^i\right\}_{t=0}^{+\infty}$ which satisfies, for all t, the following conditions:

$$\begin{aligned} q_t^i u'(y_1 - q_t^i) &= \pi q_{t+1}^i v'(y_2 + q_{t+1}^i) \\ 0 &\leq q_t^i < y_1 \end{aligned} \tag{5}$$

The dynamic equation (5) admits two stationary equilibria: a steady state without bubble $q^i = 0$ and a steady state with a positive bubble $(q^i > 0)$. A stationary stochastic bubble $q^i > 0$ is a solution of the following equation:

[4] Since the probability of the bubble to persist is lower than 1, the burst of the bubble will occur.

$$u'(y_1 - q^i) \quad = \quad \pi v'(y_2 + q^i) \tag{6}$$

Lemma 1 *(Weil 1987) Let $\bar{r}$ be the autarky implicit interest rate such that $1 + \bar{r} = u'(y_1)/v'(y_2)$. Under Assumption 1, there exists a unique stationary stochastic equilibrium $q^{i*} \in (0, y_1)$ if and only if $\pi > 1 + \bar{r}$.*

Proof. See Appendix.

A stochastic stationary bubble exists if and only if the bubble is not too risky ($\pi > 1 + \bar{r}$). This also means that when the economy of country i without bubble is dynamically efficient, that is $\bar{r} \geq 0$, no stationary stochastic bubble can exist.[5]

The global dynamics of the closed economy of country i is given by the dynamic equation (5).[6]

Lemma 2 *(Weil 1987) Under Assumption* 1*, we obtain the following conditions:*

1. *When $\pi \leq 1 + \bar{r}$, there exists no stochastic bubble.*
2. *When $\pi > 1 + \bar{r}$, then there exists a stochastic bubble such that:*
   - *if $q_0^i < q^{i*}$, then $q_t^i \to 0$ as $t \to +\infty$;*
   - *if $q_0^i = q^{i*}$, then $q_t^i = q^{i*}$ $\forall$ t;*
   - *if $q_0^i > q^{i*}$, the resulting paths are infeasible.*

Proof. See Appendix.

Since there are no connections between countries, the economic dynamics of country i before and after the burst of country j bubble remains the same, namely the dynamics of an economy with a bubble. Suppose that at time $t = 0$ agents living in Home country expects $q_0 = q^{H*}$, they will save just enough to keep constant the value of Home bubble regardless of the burst of Foreign bubble. Interdependence between asset markets are necessary for contagion to arise.

In the next section, we describe the open economy, and study the transmission of a bubble burst arising in one country to another country.

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[5] Since countries are perfectly identical, both economies without bubble are either dynamically efficient or inefficient.

[6] As the left-hand side of (5) is a strictly increasing and continuous function of q_t^i under Assumption 1, we can rewrite the dynamics of the model as follows:

$$q_t^i \quad = \quad f(q_{t+1}^i) \tag{7}$$

$$0 \quad < \quad q_t^i < y_1 \tag{8}$$

(36) depicts a backward perfect foresight dynamics which is well defined. f is continuously differentiable under our assumptions.

3 Open Economy Framework

Financial markets are now open to international trade without any transaction costs. In contrast to Section 2, households living in country $i = \{H, F\}$ can hold a diversified portfolio of Home and Foreign assets, and each country issues its own asset in a fixed amount normalized to 2.[7] Our two-country overlapping generations model is a one-good pure exchange economy with two assets. Furthermore, information are symmetric and beliefs are homogeneous across countries. If a sunspot occurs in country i, all agents perfectly receive this information and believe that the bubble issued by country i bursts.

The representative agent of a generation born at period t living in country $i = \{H, F\}$ faces the following problem:

$$\max_{c_t^i, d_{t+1}^i \geq 0} \quad \mathbb{E}_t \left[u(c_t^i) + v(d_{t+1}^i) \right] \tag{9}$$

$$s.t. \quad c_t^i + Q_t^H x_t^{i,H} + Q_t^F x_t^{i,F} = y_1 \tag{10}$$

$$d_{t+1}^i = y_2 + Q_{t+1}^H x_t^{i,H} + Q_{t+1}^F x_t^{i,F} \tag{11}$$

where $x_t^{i,j}$ $(i, j = \{H, F\})$ is the demand for country j asset of an individual living in country i.

Since countries are perfectly identical with respect preferences, endowments and population size, and agents share same beliefs in asset prices, we can claim that a household born in country i and a household born in country j with $i \neq j$ hold the same portfolio, therefore $x_t^{H,j} = x_t^{F,j} = x_t^j$.

Along the open economy equilibrium, the market clearing conditions for Home asset and Foreign asset at period t are such that:

$$x_t^{H,H} + x_t^{F,H} = 2 \text{ and } x_t^{H,F} + x_t^{F,F} = 2 \tag{12}$$

This implies $x_t^{H,j} = x_t^{F,j} = x_t^j = 1$. The world goods market clearing conditions at period t is satisfied by the Walras' law.

For the rest of the paper, we analyze two types of world economy. In the first world economy, the bubble on Home asset is immune against sunspots; whereas the bubble on Foreign asset is conditional to the realization of sunspot. In this situation, agents have full confidence in Home asset bubble and a partial one in Foreign asset bubble. In the second world economy, both bubbles are not immune against sunspots.

3.1 Home country immune against sunspots

In this section we are interested in how the burst of Foreign bubble affects the value of Home bubble when sunspots only occur in Foreign country. Only the existence of Foreign asset bubble is conditional to the realization of a sunspot. Agents in both countries believe that Foreign bubble persists $Q_t^F = q_t^f > 0$ when no sunspot occurs (with probability π), and bursts $Q_t^F = 0$ when a sunspot occurs in Foreign country (with probability $1 - \pi$). In contrast to the closed economy studied in Section 2, after the burst of Foreign bubble, all agents living in Foreign or in Home

[7] By normalizing the supply of assets to 2, the dynamics of the world economy after the burst of one of the two bubbles coincides with the dynamics studied in Section 2. If we normalize to 1, we obtain qualitatively the same results.

country can save through Home asset. Therefore, the dynamics of the economy after the crash of Foreign bubble is the same as in an economy with one asset and without uncertainty. We will show that the sunspot affecting Foreign bubble is spread to Home bubble through the portfolio of agents due to the financial integration.

Home and Foreign bubble processes at each period t are summarized by the following equations:

$$\begin{cases} Q_t^F = q_t^F > 0 \text{ with probability } \pi, Q_t^F = 0 \text{ with probability } 1-\pi; \\ \\ Q_t^H = \bar{q}_t^H \geq 0 \text{ if } Q_t^F = q_t^F > 0, Q_t^H = \underline{q}_t^H \geq 0 \text{ if } Q_t^F = 0. \end{cases} \tag{13}$$

In this framework, there are only two states of nature in the world economy such that:

$$\begin{tikzcd}[column sep=small] & \bullet\ (Q_t^F = q_t^F > 0, Q_t^H = \bar{q}_t^H \geq 0) \text{ (No sunspot in Foreing country)} \\ (Q_t^F, Q_t^H) \arrow[ur, "\pi"] \arrow[dr, "1-\pi"'] & \\ & \bullet\ (Q_t^f = 0, Q_t^H = \underline{q}_t^H \geq 0) \text{ (Sunspot in Foreign country)} \end{tikzcd}$$

When Foreign bubble is not valued in period t, $Q_t^F = 0$, then the optimal behavior of an agent born in country i is summarized by the following equation:

$$\frac{u'\left(y_1 - \underline{q}_t^H x_t^{i,H}\right)}{v'\left(y_2 + \underline{q}_{t+1}^H x_t^{i,H}\right)} = \frac{\underline{q}_{t+1}^H}{\underline{q}_t^H} \tag{14}$$

$$\tag{15}$$

This intertemporal trade-off is similar to the one found in the monetary model by Samuelson (1958). In this case, the economy of country i behaves as an economy with one asset and without uncertainty along an equilibrium.

When Foreign bubble is valued in period t, $Q_t^F = q_t^F > 0$, the optimal behavior of an agent born in country i is given by the intertemporal trade-off and the no-arbitrage condition between both bubbles:

$$\frac{u'\left(y_1 - q_t^F x_t^{i,F} - \bar{q}_t^H x_t^{i,H}\right)}{v'(y_2 + q_{t+1}^F x_t^{i,F} + \bar{q}_{t+1}^H x_t^{i,H})} = \pi \frac{q_{t+1}^F}{q_t^F} \tag{16}$$

$$\pi \frac{q_{t+1}^F}{q_t^F} = \pi \frac{\bar{q}_{t+1}^H}{\bar{q}_t^H} + (1-\pi) \frac{\underline{q}_{t+1}^H}{\bar{q}_t^H} \frac{v'\left(y_2 + \underline{q}_{t+1}^H x_t^{i,H}\right)}{v'\left(y_2 + q_{t+1}^F x_t^{i,F} + \bar{q}_{t+1}^H x_t^{i,H}\right)} \tag{17}$$

From the no-arbitrage condition (17), we deduce that agents invest in both assets as long as the expected marginal return on Foreign bubble is equal to the expected marginal return on Home bubble weighted by the marginal utilities.

Since we are interested in the effect of Foreign bubble burst on the value of Home bubble, we restrict our focus on the case where Foreign bubble is valued at time t. We solve by backward induction to determine the equilibrium with Foreign and Home bubble. After Foreign bubble crash, the economy of country i moves to the equilibrium with one asset and without uncertainty satisfying the following dynamic equation:

$$\frac{u'\left(y_1 - \underline{q}_t^H\right)}{v'\left(y_2 + \underline{q}_{t+1}^H\right)} = \frac{\underline{q}_{t+1}^H}{\underline{q}_t^H} \tag{18}$$

$$0 \leq \underline{q}_t^H \leq y_1 \tag{19}$$

Before the crash of Foreign bubble, an equilibrium is a sequence $\left\{q_t^F, \overline{q}_t^H\right\}_{t=0}^{+\infty}$ which satisfies for all t the following conditions:

$$\frac{u'\left(y_1 - q_t^F - \overline{q}_t^H\right)}{v'(y_2 + q_{t+1}^F + \overline{q}_{t+1}^H)} = \pi \frac{q_{t+1}^F}{q_t^F} \tag{20}$$

$$\pi\left(\frac{q_{t+1}^F}{q_t^F} - \frac{\overline{q}_{t+1}^H}{\overline{q}_t^H}\right) = (1-\pi)\frac{\underline{q}_{t+1}^H}{\overline{q}_t^H}\frac{v'\left(y_2 + \underline{q}_{t+1}^H\right)}{v'\left(y_2 + q_{t+1}^F + \overline{q}_{t+1}^H\right)} \tag{21}$$

where $\underline{q}_{t+1}^H$is an equilibirum value of Home bubble satisfying equation (18).

To highlight the effect of Foreign bubble burst on the dynamics of Home bubble, we simplify the analysis considering only the stationary equilibrium. In this equilibrium, Foreign and Home bubble are constant before the burst if Foreign bubble such that $q_t^F = q^F$ and $\overline{q}_t^H = \overline{q}^H$ for all t.

We can show from (21) that a stationary equilibrium exists before the crash of Foreign bubble only if $\underline{q}_{t+1}^H = 0$. As countries are symmetric in terms of fundamentals, the expected return on Foreign bubble is equal to the the expected return on Home bubble at the stationary equilibrium. Therefore, the expected marginal gain to invest in Foreign bubble is zero. Because of the no-arbitrage condition, the marginal loss to invest in Foreign bubble must be zero. This implies that the value of Home bubble after the burst of Foreign bubble is zero. Otherwise, an arbitrage opportunity would exist.

Let $Q = q^F + \overline{q}^H$. Given $\underline{q}_{t+1}^H = 0$, a stationary equilibrium Q^* (with $q^F > 0$ and $\overline{q}^H > 0$ is a solution of the following equation:

$$u'(y_1 - Q) = \pi v'(y_2 + Q) \tag{22}$$

$$\tag{23}$$

Proposition 1 *Under Assumption* 1 *and given* $\underline{q}_{t+1}^H = 0$*, a stationary stochastic equilibrium* $Q^* \in (0, y_1)$ *with stochastic bubbles exists if and only if* $\pi > 1 + \bar{r}$*.*

Proof. See Appendix.

Proposition 1 means that upon to the random date on which Foreign bubble bursts, both countries experience a stochastic stationary bubble provided that bubbles are sufficiently safe. This equilibrium may occur when the economy without bubble are dynamically inefficient. The condition for existence of stochastic stationary bubbles in open economy is the same as for the existence a stochastic bubble in closed economy. As corollary to Proposition 1 we obtain the following:

Corollary 1 *Let $\pi > 1 + \bar{r}$. Home bubble jumps to zero and does not reappear in the future after the burst of Foreign bubble.*

When Foreign bubble explodes, Home bubble explodes as well in the same time. In other words, after the realization of a sunspot in Foreign country, no bubbles persist in both countries. Figure 1 give a qualitative illustration of Proposition 1 and Corollary 1. We assume that the economy experiences Home and Foreign bubble and is in the stationary equilibrium before period T. At period T, a sunspot occurs in Foreign country implying the burst of Foreign bubble. At the same period T, Home bubble jumps to zero.

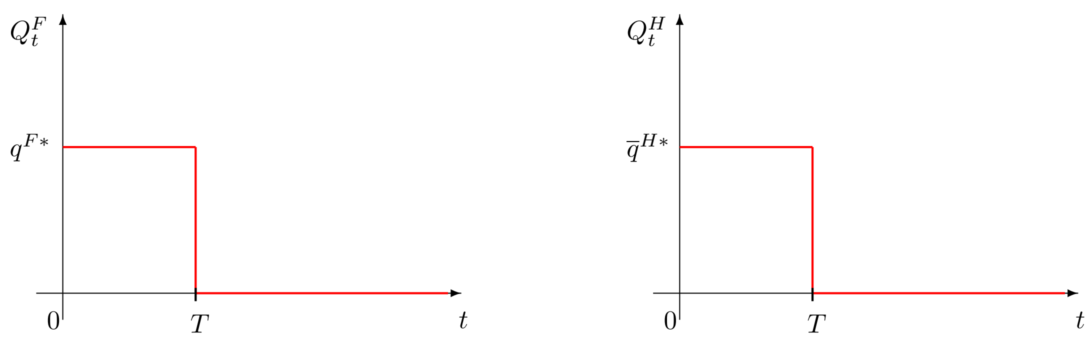


Figure 1:

In this framework, we show that even though the realization of sunspots is specific to Foreign country, a sunspot is spread to Home country through the portfolio choice of agents under financial market integration. Following the burst of Foreign bubble, agents lose confidence in Home bubble. All old agents would like to sell Home bubble and no young agents would like to buy it. Thus, Home bubble bursts as well.

In the next section, we study the second case in which Home and Foreign country are not immune against a sunspot shock.

3.2 Home and Foreign countries not immune against sunspots

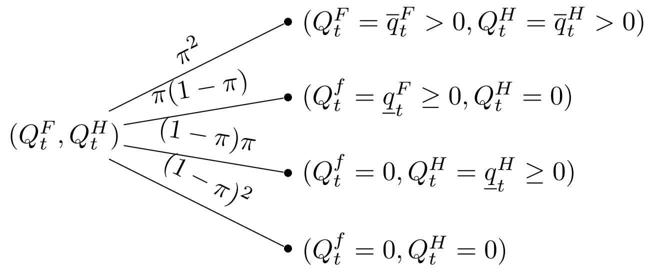
In this section, we study the effect of Foreign bubble burst on Home bubble when both bubbles are affected by the realization of a country-specific sunspot. More precisely, the existence of Home and Foreign bubble is conditional to the realization of different random events: Home bubble is conditional to sunspot H and Foreign bubble to sunspot F. To highlight the role of financial integration in the transmission, we suppose that sunspot H and sunspot F are two extrinsic independent and identical distributed random variables. Sunspot H and sunspot F have same probability $1 - \pi$ to appear. In other words, agents as much confidence in Home bubble as in Foreign bubble.[8] The price processes at each period t are summarized by the following equations:

[8] If the two random variables are not identical distributed, there cannot exist an equilibrium. Suppose that Foreign bubble has a higher probability to persist than Home bubble, but Foreign bubble bursts in the first place. In such a situation, all agents who hold Home bubble would like to sell it and no one would like to buy it. Therefore, there would be an excess supply, an equilibrium with positive price could not exist.

$$
\begin{cases}
Q_t^i = q_t^i > 0 \text{ with probability } \pi, = 0 \text{ with probability } 1-\pi; \\
\\
Q_t^j = \bar{q}_t^j > 0 \text{ if } Q_t^i = q_t^i > 0, Q_t^j = \underline{q}_t^j \geq 0 \text{ if } Q_t^i = 0.
\end{cases}
\tag{24}
$$

where $i, j = \{H, F\}$ and $i \neq j$.

In contrast to Section 3.1, there are now four states of nature in the world: (no sunspot H, no sunspot F), (sunspot H, no sunspot F), (no sunspot H, sunspot F) and (sunspot H, sunspot F).



When none of both bubbles are valued in period t, $Q_t^F = 0$ and $Q_t^H = 0$, the economy of country i is in the autarky regime. There are neither international trades nor intergenerational trades.

When only one of these two bubbles is valued at period t, $Q_t^i = 0$ and $Q_t^j = \underline{q}_t^j > 0$ with $i \neq j$, the optimal behaviour of an agent born in Country i is given by the following equation:

$$
\frac{u'\left(y_1 - \underline{q}_t^j x_t^{i,j}\right)}{v'(y_2 + \underline{q}_{t+1}^j x_t^{i,j})} = \pi \frac{\underline{q}_{t+1}^j}{\underline{q}_t^j} \tag{25}
$$

Since the economy of corresponds to a single-good exchange economy with one asset in this case, we find the same intertemporal trade-off as the closed economy studied in Section 2.

Lastly, when both bubbles are valued at period t $Q_t^H = \bar{q}_t^H > 0$ and $Q_t^F = \bar{q}_t^F > 0$, the optimal behaviour of an agent born in Country i is summarized by:

$$
\frac{u'\left(y_1 - \bar{q}_t^H x_t^{i,H} - \bar{q}_t^F x_t^{i,F}\right)}{v'\left(y_2 + \bar{q}_{t+1}^H x_t^{i,H} + \bar{q}_{t+1}^F x_t^{i,F}\right)} = \pi \left[\pi \frac{\bar{q}_{t+1}^H}{\bar{q}_t^H} + (1-\pi) \frac{\underline{q}_{t+1}^H}{\bar{q}_t^H} \frac{v'\left(y_2 + \underline{q}_{t+1}^H x_t^{i,H}\right)}{v'(y_2 + \bar{q}_{t+1}^H x_t^{i,H} + \bar{q}_{t+1}^F x_t^{i,F})}\right] \tag{26}
$$

$$
\begin{aligned}
&\pi \frac{\bar{q}_{t+1}^F}{\bar{q}_t^F} v'\left(y_2 + \bar{q}_{t+1}^H x_t^{i,H} + \bar{q}_{t+1}^F x_t^{i,F}\right) + (1-\pi) \frac{\underline{q}_{t+1}^F}{\bar{q}_t^F} v'\left(y_2 + \underline{q}_{t+1}^F x_t^{i,F}\right) = \\
&\quad \pi \frac{\bar{q}_{t+1}^H}{\bar{q}_t^H} v'\left(y_2 + \bar{q}_{t+1}^H x_t^{i,H} + \bar{q}_{t+1}^F x_t^{i,F}\right) + (1-\pi) \frac{\underline{q}_{t+1}^H}{\bar{q}_t^H} v'\left(y_2 + \underline{q}_{t+1}^H x_t^{i,H}\right)
\end{aligned}
\tag{27}
$$

Since both countries are not immune against sunspots, we obtain a modified intertemporal trade-off and a modified the no-arbitrage condition between Foreign and Home bubbles compared to the case in which only Foreign country is affected by sunspots.

To simplify the analysis, we consider that a sunspot occur in Foreign country in first place. Therefore, the bubble issued by Foreign country bursts before the bubble of Home country.[9].

To highlight the effect of the burst of Foreign bubble on the value of Home bubble, we restrict our focus on the case in which both bubbles are valued at time t, $Q_t^F = \overline{q}_t^F > 0, Q_t^H = \overline{q}_t^H > 0$. We solve by backward induction to determine the equilibrium with Foreign and Home bubble.

After the burst of Foreign bubble, the economy of each country moves to the equilibirum with one asset satisfying the following dynamic equation:

$$\frac{u'\left(y_1 - \underline{q}_t^H\right)}{v'\left(y_2 + \underline{q}_{t+1}^H\right)} = \pi \frac{\underline{q}_{t+1}^H}{\underline{q}_t^H} \tag{28}$$

$$0 \leq \underline{q}_t^H \leq y_1 \tag{29}$$

The economy behaves as the closed economy studied in Section 2 after the collapse of Foreign bubble. From Lemma 2, we know that a stationary equilibrium $\underline{q}^{H*} \in (0, y_1)$ exists if $\pi > 1 + \bar{r}$.

Before the crash of Foreign bubble, an equilibrium is a sequence $\left\{\overline{q}_t^F, \overline{q}_t^H\right\}_{t=0}^{+\infty}$ which satisfies for all t the following conditions:

$$\frac{u'\left(y_1 - \bar{q}_t^H - \bar{q}_t^F\right)}{v'(y_2 + \bar{q}_{t+1}^H + \bar{q}_{t+1}^F)} = \pi \left[\pi \frac{\bar{q}_{t+1}^H}{\bar{q}_t^H} + (1-\pi) \frac{\underline{q}_{t+1}^H}{\bar{q}_t^H} \frac{v'\left(y_2 + \underline{q}_{t+1}^H\right)}{v'\left(y_2 + \bar{q}_{t+1}^H + \bar{q}_{t+1}^F\right)}\right] \tag{30}$$

$$\pi \left(\frac{\bar{q}_{t+1}^F}{\bar{q}_t^F} - \frac{\bar{q}_{t+1}^H}{\bar{q}_t^H}\right) v'\left(y_2 + \bar{q}_{t+1}^H + \bar{q}_{t+1}^F\right) = -(1-\pi)\left[\frac{\underline{q}_{t+1}^F}{\bar{q}_t^F} v'\left(y_2 + \underline{q}_{t+1}^F\right) - \frac{\underline{q}_{t+1}^H}{\bar{q}_t^H} v'\left(y_2 + \underline{q}_{t+1}^H\right)\right] \tag{31}$$

where $\underline{q}_{t+1}^F$ and $\underline{q}_{t+1}^H$ are respectively an equilibirum value of Foreign bubble and Home bubble satisfying equation (28). Note that in period t, agents do not known which bubble will burst in period $t+1$. Hence, the presence of $\underline{q}_{t+1}^F$ in (31). For the rest of the analysis, we take $\underline{q}_{t+1}^H = a \in (0, \underline{q}^{H*})$ and $\underline{q}_{t+1}^F = b \in (0, \underline{q}^{F*})$ as given. Note that as countries are perfectly identical $\underline{q}^{H*} = \underline{q}^{F*} = \underline{q}^*$.

We study now the existence of a stationary stochastic equilibrium with Foreign and Home bubble before the crash of Foreign bubble.

A stationary stochastic equilibrium $(\bar{q}^{F*}, \bar{q}^{H*})$ with $\bar{q}^{F*} > 0$ and $\bar{q}^{H*} > 0$ is a solution of the following equation:

$$u'\left(y_1 - (\bar{q}^F + \bar{q}^H)\right) = \pi \left[\pi v'(y_2 + \bar{q}^F + \bar{q}^H) + (1-\pi) \frac{a}{\bar{q}^H} v'\left(y_2 + a\right)\right] \tag{32}$$

$$\frac{b}{\bar{q}^F} v'\left(y_2 + b\right) = \frac{a}{\bar{q}^H} v'\left(y_2 + a\right) \tag{33}$$

Proposition 2 *Let $\bar{Q}^* = \bar{q}^{F*} + \bar{q}^{H*}$. Under Assumption 1, the following generically holds before the burst of country $i = \{H, F\}$ bubble:*

[9] Since countries are identical, we obtain same result, if we assume that Home bubble is the first to explode.

1. *If $\pi \leq 1+\bar{r}$, then there does not exist a stationary equilibrium with stochastic bubbles.*

2. *If $1+\bar{r} < \pi \leq (1+\bar{r})^{1/2}$, then a unique stationary equilibrium $\bar{Q}^* \in (0, y_1)$ $\forall$ $(a,b) \in]0, \underline{q}^*]$ exists with stochastic bubbles $\bar{q}^{F*} = \bar{Q}^* bv'(y_2+b)/\left[av'(y_2+a)+bv'(y_2+b)\right]$ and $\bar{q}^{H*} = \bar{Q}^* av'(y_2+a)/\left[av'(y_2+a)+bv'(y_2+b)\right]$.*

3. *If $\pi > (1+\bar{r})^{1/2}$, then a unique stationary equilibrium $\bar{Q}^* \in (0, y_1)$ $\forall$ $(a,b) \in [0, \underline{q}^*]$ exists with stochastic bubbles.*

Proof. See Appendix.

Proposition 2 shows that if agents expect a positive value of Home bubble after the burst of Foreign bubble, then both countries can experience a stochastic stationary bubble, as long as the economy without bubble are dynamically inefficient. We find the similar requirement for the existence of a stationary equilibrium as in the closed economy (Weil (1987)). If agents expect that the value of Home bubble is equal to zero after the burst of Foreign bubble, then the scope for the existence of a stochastic stationary equilibrium is reduced. Only safe bubble can exist. This equilibrium is likely to occur in a dynamically inefficient economy only if the probability for Foreign and Home bubbles to persist is sufficiently closed to 1.

To highlight the impact of the burst of Foreign bubble on the existence of Home bubble, we assume that the economy experiences Home and Foreign bubble and is in the stationary equilibrium before period T. At period T, a sunspot occurs in Foreign country implying the burst of Foreign bubble.

Proposition 3 *Under Assumption 1, the stationary value of Home bubble $\bar{q}^{H*}$ before the burst of Foreign bubble is strictly lower than $\underline{q}^{H*}$, where $\underline{q}^{H*}$ is the stationary value of Home bubble after the burst of Foreign bubble.*

Proof. See Appendix.

Proposition 3 relies on an argument of market size.

For the rest of the paper, we restrict our focus only on safe bubble assuming that:

Assumption 2 $\pi > (1+\bar{r})^{1/2}$

When $\bar{q}^{H*} = \bar{q}^{F*}$ before the crash of Foreign bubble, three cases emerge according to agents' expectations when both countries are subject to sunspots in contrast to section 3.1.

From the analysis of Home bubble dynamics after Foreign bubble crash (see Lemma 2) and Proposition 2, we derive three corollaries.

Corollary 2 *Under Assumptions 1 and 2, if $a = \underline{q}_T^H = 0$, then Home bubble jumps to zero after the burst of Foreign bubble and does not reappear in the future.*

Following the burst of Foreign bubble in period T, all agents expect that the value of Home bubble will be zero in period T, and beliefs are self-fulfilling. This result is similar to the one fund in Section 3.1 when Home country is immune against sunspots (see Figure 1 for a qualitative illustration).

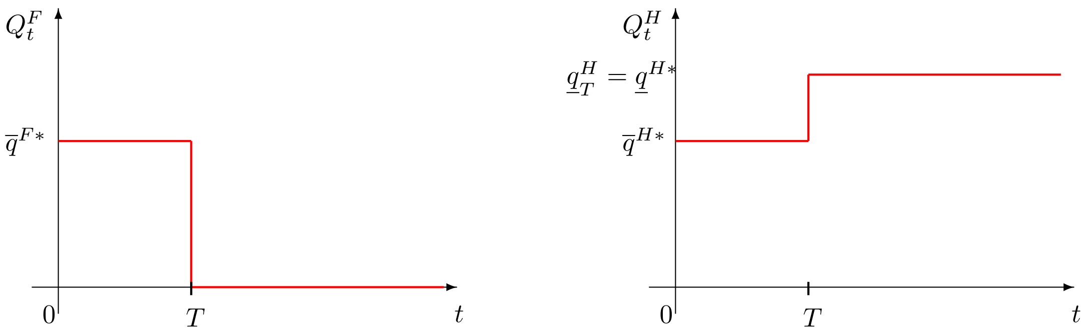


Figure 2:

Corollary 3 *Under Assumptions 1 and 2, if $a = \underline{q}_T^H = \underline{q}^{H*}$, then Home bubble jumps to $\underline{q}^{H*}$ after the burst of Foreign bubble, and stays at this value until the occurrence of a sunspot in Home country.*

If agents expect that the value of Home bubble immediately after the burst of Foreign bubble in period T $\underline{q}_T^H$ is equal to the stationary value $\underline{q}^{H*}$, then they save enough to keep the value of Home bubble constant until the burst of Home bubble. In this case, the effect of Foreign bubble burst is positive (see Figure 2). After the burst of Foreign bubble, young agents can save only through Home bubble. Savings in terms of Home bubble increases. Therefore, the value of Home bubble after Foreign bubble crash ($\underline{q}_T^H = \underline{q}^{H*}$) is greater than before the burst ($\overline{q}^{H*}$).**[Note that the probability for this case to occur is small. Since the steady state after Foreign bubble burst is unstable, the probability for $\underline{q}_T^H$ to be equal to $\underline{q}^{i*}$ is small.]** The next corollary depicts the last case:

Corollary 4 *Under Assumptions 1 and 2, if $a = \underline{q}_T^H \in]0, \underline{q}^{H*}[$, then Home bubble jumps to $\underline{q}_T^H$ after the burst of Foreign bubble, and converges to zero in the long run.*

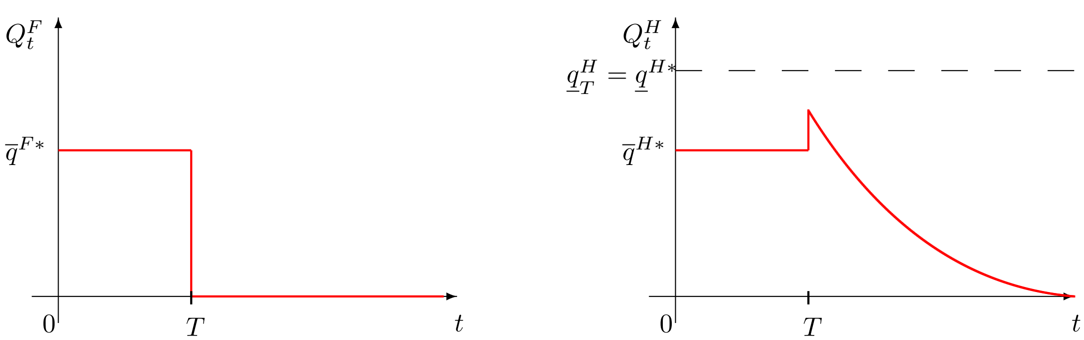


Figure 3:

In the last case, agents expect that the value of Home bubble immediately after the crash of Foreign bubble $\underline{q}_T^H$ is positive but less than the stationary value $\underline{q}^{H*}$. The return of Home bubble falls. Agents flee financial market and save less, the Home bubble collapse (see Figure 3 for a qualitative illustration). The effect of Foreign bubble burst is negative in long term. Home bubble tends to collapse after the Foreign bubble crash even before the occurrence of a sunspot in Home country.

In contrast to Section 3.1, the burst of Foreign bubble does not necessarily imply the immediate crash of Home bubble. When both countries are subject to sunspots, agents internalize the risk of Home bubble burst, and thus continue to invest in Home bubble after the crash of Foreign bubble.

[To be developped].

When $\bar{q}^{H*} > \bar{q}^{F*}$ before the crash of Foreign bubble, Corollaries 2 and 3 hold.

When $\bar{q}^{H*} < \bar{q}^{F*}$ before the crash of Foreign bubble, Corollary 3 hold. (See Appendix)

4 Concluding remarks

In this paper, we develop a simple neoclassical framework to analyze the international transmission of bubble crashes. We extend Weil's (1987) overlapping generations model in which the burst of the bubble is based on self-fulfilling beliefs (sunspot) to a two-country model with information symmetry and homogeneous beliefs. Each country issues its own bubble, and each agents can hold a portfolios of both assets. We show that even though the realization of sunspots is specific to one country, the sunspot is spread to other country through the portfolio choice of agents. When only the foreign country is subject to sunspots, the burst of the foreign bubble generates a change in confidence in the domestic bubble involving its immediate collapse. When both countries are subject to sunspots, agents internalize the risk of the domestic bubble burst. Therefore, the domestic bubble can persist just after the burst of Foreign bubble, but may deflate over time.

Our framework can be considered as a first step for future research. For instance, we could extend it to analyze the impact of government policies (like transaction costs) on the international transmission of bubble crashes.

5 Appendix

Proof of Lemma 1

This proof is similar as the one given by Weil (1987). A steady state q^i is a solution of $A(q^i) = B(q^i)$, with:

$$A\left(q^i\right) \equiv u'\left(y_1 - q^i\right) \tag{34}$$

$$B\left(q^i\right) \equiv \pi v'\left(y_2 + q^i\right) \tag{35}$$

One has $A'\left(q^i\right) > 0$ and $B'\left(q^i\right) < 0$. As $c > 0$ implies $q^i < y_1$, all the stationary solutions q^i belong to $(0, y_1)$.

To prove the existence of a stationary solution b, we use the continuity of $A\left(q^i\right)$ and $B\left(q^i\right)$.

$$\begin{aligned}
\lim_{q^i \to 0} A\left(q^i\right) &= u'\left(y_1\right) \\
\lim_{q^i \to 0} B\left(q^i\right) &= \pi v'\left(y_2\right) \\
\lim_{q^i \to y_1} A\left(q^i\right) &= u'\left(0\right) = +\infty \\
\lim_{q^i \to y_1} B\left(q^i\right) &= \pi v'\left(y_2 + y_1\right)
\end{aligned}$$

We have $\lim_{q^i \to y_1} A\left(q^i\right) > \lim_{q^i \to y_1} B\left(q^i\right)$. If $\pi \leq u'(y_1)/\left[v'(y_2)\right]$, then $\lim_{q^i \to 0} A\left(q^i\right) > \lim_{q^i \to 0} B\left(q^i\right)$. No stochastic bubble steady state exists. If $\pi > u'(y_1)/\left[v'(y_2)\right]$, then $\lim_{q^i \to 0} A\left(q^i\right) < \lim_{q^i \to 0} B\left(q^i\right)$. A stochastic bubble stationary equilibrium $q^{i*} \in (0, y_1)$ exists. ■

Proof of Lemma 2

As the left-hand side of (5) is a strictly increasing and continuous function of q_t^i, we can rewrite the dynamics of the model as follows:

$$\begin{aligned}
q_t^i &= f(q_{t+1}^i) \qquad (36) \\
0 &< q_t^i < y_1 \qquad (37)
\end{aligned}$$

(36) depicts a backward perfect foresight dynamics which is well defined. f is continuously differentiable and $f(0) = 0$ under Assumption 1. Moreover, let $\varepsilon_v(d) \equiv -v''(d)d/v'(d)$ and $\varepsilon_u(c) \equiv -u''(c)c/u'(c)$, $f'(q^*)$ is given by:

$$f'(q^{i*}) = \frac{1 - \epsilon_v(d^*)q^*/\left(y_2 + q^*\right)}{1 + \epsilon_u(c^*)q^*/\left(y_1 - q^*\right)} \in (0, 1) \qquad (38)$$

$$\text{with } c^* = y_1 - q^{i*} \text{ and } d^* = y_2 + q^{i*} \qquad (39)$$

Hence, $f(q^i) > q^i$ when $q^i \in (0, q^{i*})$, $f(q^i) = q^i$ when $q^i = q^{i*}$, and $f(q^i) < q^i$ when $q^i \in (q^{i*}, y_1)$.
Under Assumption 1, the steady state q^{i*} is unstable in the forward dynamics.
Furthermore, when $q_0 > q^{i*}$, there exists a date $t + s$ with $s > 0$ during which $q_{t+s}^i > y_1$. This implies $c_{t+s} < 0$ which is infeasible. ■

Proof of Proposition 1

The proof is equivalent to the proof of Lemma 1. ■

Proof of Proposition 2

To prove the existence of a stochastic stationary equilibrium $(\overline{q}^F, \overline{q}^H)$ before the burst of country $i = \{H, F\}$ bubble, we have to solve by backward induction: first, we analyze the economy of country i after the crash of country j, then before the crash.

After the burst of country j bubble, the economy is equivalent to the one studied in Section 2. From Weil (1987), we deduce that when $\pi \leq u'(y_1)/v'(y_2) \equiv 1 + \bar{r}$, no stationary stochastic country i bubble $\underline{q}^{i*}$ exists after the burst of country j bubble under our Assumption 1. When $\pi > 1 + \bar{r}$, a stationary stochastic Home bubble exists in the economy after the burst of Foreign bubble under Assumption 1. The dynamics of Home bubble after the burst of Foreign bubble is given in Lemma 2.

Now, we analyze the existence of the stochastic stationary equilibrium $(\bar{q}^F, \bar{q}^H)$ before the burst of Foreign bubble.

A stationary stochastic equilibrium $(\bar{q}^{F*}, \bar{q}^{H*})$ with $\bar{q}^{F*} > 0$ and $\bar{q}^{H*} > 0$ is a solution of the following equation:

$$u'\left(y_1 - (\bar{q}^F + \bar{q}^H)\right) = \pi\left[\pi v'(y_2 + \bar{q}^F + \bar{q}^H) + (1-\pi)\frac{a}{\bar{q}^H} v'(y_2 + a)\right] \tag{40}$$

$$\frac{b}{\bar{q}^F} v'(y_2 + b) = \frac{a}{\bar{q}^H} v'(y_2 + a) \tag{41}$$

where $a = q_{t+1}^H \in (0, \underline{q}^{H*})$ and $b = q_{t+1}^F \in (0, \underline{q}^{F*})$ are respectively the equilibrium value after the bubble crash of country F and after the bubble crash of country H.

Let $\bar{Q} = \bar{q}^F + \bar{q}^H$. From 41, we get:

$$\bar{q}^H = \bar{Q}\frac{av'(y_2 + a)}{av'(y_2 + a) + bv'(y_2 + b)} \tag{42}$$

$$\bar{q}^F = \bar{Q}\frac{bv'(y_2 + b)}{av'(y_2 + a) + bv'(y_2 + b)} \tag{43}$$

A stochastic stationary equilibrium $\bar{Q}$ is a solution of the following equation:

$$u'\left(y_1 - \bar{Q}\right) = \pi\left[\pi v'\left(y_2 + \bar{Q}\right) + (1-\pi)\frac{av'(y_2 + a) + bv'(y_2 + b)}{\bar{Q}}\right] \tag{44}$$

Let $G(\bar{Q}) \equiv u'\left(y_1 - \bar{Q}\right)$ and $H(\bar{Q}) \equiv \pi\left\{\pi v'\left(y_2 + \bar{Q}\right) + (1-\pi)\left[av'(y_2 + a) + bv'(y_2 + b)\right]/\bar{Q}\right\}$. One has $G'\left(\bar{Q}\right) > 0$ and $H'\left(\bar{Q}\right) < 0$. As $c = y_1 - \bar{Q} > 0$ implies $\bar{Q} < y_1$, all the stationary solutions $\bar{Q}$ belong to $(0, y_1)$. To prove the existence of a stationary solution $\bar{Q}$, we use the continuity of $G\left(\bar{Q}\right)$ and $H\left(\bar{Q}\right)$.

We consider first the case in which $\pi \leq 1 + \bar{r}$. When $\pi \leq 1 + \bar{r}$, we know from Lemma 2 that $a = 0$ and $b = 0$. Therefore,

$$\begin{aligned}
\lim_{\bar{Q}\to 0} G\left(\bar{Q}\right) &= u'\left(y_1\right) \\
\lim_{\bar{Q}\to 0} H\left(\bar{Q}\right) &= \pi^2 v'\left(y_2\right) \\
\lim_{\bar{Q}\to y_1} G\left(\bar{Q}\right) &= u'\left(0\right) = +\infty \\
\lim_{\bar{Q}\to y_1} H\left(\bar{Q}\right) &= \pi^2 v'\left(y_2 + y_1\right)
\end{aligned}$$

We have $\lim_{\bar{Q}\to y_1} G\left(\bar{Q}\right) > \lim_{\bar{Q}\to y_1} H\left(\bar{Q}\right)$. When $\pi^2 \leq 1 + \bar{r}$, $\lim_{\bar{Q}\to 0} G\left(\bar{Q}\right) > \lim_{\bar{Q}\to 0} H\left(\bar{Q}\right)$. Therefore, if $\pi^2 \leq 1 + \bar{r}$, then no stochastic stationary bubble exists under Assumption 1. When $\pi^2 > 1 + \bar{r}$, $\lim_{\bar{Q}\to 0} G\left(\bar{Q}\right) < \lim_{\bar{Q}\to 0} H\left(\bar{Q}\right)$. Thus, if $\pi^2 > 1 + \bar{r}$, then a unique stochastic stationary bubble $\bar{Q}^* \in (0, y_1)$ could exist under Assumption 1. However, as $\pi < 1$ and $\pi \leq 1+\bar{r}$, the condition $\pi^2 > 1+\bar{r}$ does not hold. Therefore, no stochastic stationary bubble exists in the open economy when $\pi \leq 1+\bar{r}$.

Next, we analyze the case where $\pi > 1+\bar{r}$. First, when $a = 0$ ($b = 0$), then stationary stochastic bubbles $\bar{q}^{F*} > 0$ and $\bar{q}^{H*} > 0$ exist if and only if $b = 0$ ($a = 0$). Furthermore, we know that when $a = 0$ and $b = 0$, a stochastic stationary equilibrium exists if and only if $\pi^2 > 1+\bar{r}$.

Now, we analyze the existence of a stochastic stationary equilibrium when $(a, b) \in]0, \underline{q}^*]$. First, we can note that $a < \bar{Q}$ and $b < \bar{Q}$. We make a proof by contradiction. Suppose that $a \geq \bar{Q}$. As $\bar{Q} > \bar{q}^H$, we deduce from (40) that a stationary equilibrium $\bar{Q}^*$ satisfies the following inequality:

$$u'\left(y_1 - \bar{Q}^*\right) > \pi[\pi v'(y_2 + \bar{Q}^*) + (1-\pi)v'(y_2 + a)]$$

Because of utility concavity, we obtain:

$$u'\left(y_1 - \bar{Q}^*\right) > \pi v'(y_2 + a)$$

Since $a \leq \underline{q}^{H*}$, we get

$$u'\left(y_1 - \bar{Q}^*\right) > \pi v'(y_2 + \underline{q}^{H*})$$

We recall that after the burst of Foreign bubble, a stationary equilibrium $\underline{q}^{H*}$ is such that:

$$u'\left(y_1 - \underline{q}^{H*}\right) = \pi v'(y_2 + \underline{q}^{H*}) \tag{45}$$

Hence, $\bar{Q}^* > \underline{q}^{H*} \geq a$. This results in a contradiction. Therefore, $a < \bar{Q}$. We can provide a similar proof for $b < \bar{Q}$.

We determine the boundary values of $G(\bar{Q})$ and $H(\bar{Q})$ $\forall$ $a > 0$ and $b > 0$.

$$\begin{aligned}
\lim_{\bar{Q}\to 0} G\left(\bar{Q}\right) &= u'\left(y_1\right) \\
\lim_{\bar{Q}\to 0} H\left(\bar{Q}\right) &= \pi\left[\pi v'(y_1) + (1-\pi)\frac{av'(y_2+a) + bv'(y_2+b)}{0}\right] = +\infty \\
\lim_{\bar{Q}\to y_1} G\left(\bar{Q}\right) &= u'\left(0\right) = +\infty \\
\lim_{\bar{Q}\to y_1} H\left(\bar{Q}\right) &= \pi\left[\pi v'\left(y_2 + y_1\right) + (1-\pi)\frac{av'(y_2+a) + bv'(y_2+b)}{y_1}\right]
\end{aligned}$$

We have $\lim_{\bar{Q}\to 0} G\left(\bar{Q}\right) < \lim_{\bar{Q}\to 0} H\left(\bar{Q}\right)$ and $\lim_{\bar{Q}\to y_1} G\left(\bar{Q}\right) > \lim_{\bar{Q}\to y_1} H\left(\bar{Q}\right)$. We deduce that for given equilibrium values $a \in]0, \underline{q}^*]$ and $b \in]0, \underline{q}^*]$, there exists a unique stochastic stationary equilibrium $\bar{Q}^* \in (0, y_1)$. ■

Proof of Proposition 3

Differentiating (44) and (42) (at the stationaryy equilibirum), we obtain:

$$\frac{\partial \bar{Q}^*}{\partial a} = \frac{\pi(1-\pi)\left[v'(y_2+a)+av''(y_2+a)\right]}{-u''(y_1-\bar{Q}^*)-\pi^2 v''(y_2+\bar{Q}^*)+\pi(1-\pi)\left[av'(y_2+a)+bv'(y_2+b)\right]/\bar{Q}^{*2}} \quad (46)$$

$$\frac{\partial \bar{q}^{H*}}{\partial a} = \frac{av'(y_2+a)}{av'(y_2+a)+bv'(y_2+b)}\frac{\partial \bar{Q}^*}{\partial a}+\bar{Q}^* bv'(y_2+b)\frac{v'(y_2+a)+av''(y_2+a)}{av'(y_2+a)+bv'(y_2+b)} \quad (47)$$

Under Assumption 1, $\partial \bar{Q}^*/\partial a > 0$ and $\partial \bar{q}^{H*}/\partial a > 0$.

To prove that $\bar{q}^{H*} < \underline{q}^{H*}$, we make a proof by contradiction. Suppose that $a = \underline{q}^{H*}$ and $\underline{q}^{H*} \leq \bar{q}^{H*}$. (40) is written as follows:

$$u'\left(y_1-\bar{Q}^*\right) = \pi\left[\pi v'(y_2+\bar{Q}^*)+(1-\pi)\frac{\underline{q}^{H*}}{\bar{q}^{H*}}v'\left(y_2+\underline{q}^{H*}\right)\right] \leq \pi\left[\pi v'(y_2+\bar{Q}^*)+(1-\pi)v'\left(y_2+\underline{q}^{H*}\right)\right]$$

As $\bar{Q} > \bar{q}^H$, the concavity of the utility function implies:

$$u'\left(y_1-\bar{Q}^*\right) < \pi v'\left(y_2+\underline{q}^{H*}\right)$$

$\underline{q}^{H*}$ satisfies the following equation $u'\left(y_1-\underline{q}^{H*}\right) = \pi v'(y_2+\underline{q}^{H*})$.
Hence,

$$u'\left(y_1-\bar{Q}^*\right) < u'\left(y_1-\underline{q}^{H*}\right)$$

Because of utility concavity, we obtain $\bar{Q}^* < \underline{q}^{H*}$. Hence, $\bar{q}^{H*} < \underline{q}^{H*}$, it results in a contradiction. Therefore, when $a = \underline{q}^{H*}$, $\bar{q}^{H*} < \underline{q}^{H*}$.
Since $\partial \bar{q}^{H*}/\partial a > 0$, we deduce that $\bar{q}^{H*} < \underline{q}^{H*}$ $\forall\ a \in (0, \underline{q}^{H*})$. ■

If $\bar{q}^{H*} > \bar{q}^{F*}$ before the crash of Foreign bubble, then $0 < b < a \leq \underline{q}^{H*}$ (see (41)). Therefore, according to Lemma 2 and Proposition 2, Corollaries 2 and 3 hold.
If $\bar{q}^{H*} < \bar{q}^{F*}$ before the crash of Foreign bubble, then $0 < a < b \leq \underline{q}^{H*}$. Hence, Corollary 3 only holds.

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