Schumpeterian business cycles

(work in progress)

Filip Rozsypal

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Abstract

This paper presents an economy where business cycles and long term growth are two outcomes of the same type of shocks. I embed a multi-sector real business cycle model into a endogenous growth framework where innovating firms replace incumbent production firms. The only source of uncertainty is an imperfectly observed quality of innovation projects. The amplification comes from the complementarity of final goods; successful innovation in one sector increases demand for the output of other sectors. Higher profits motivate higher innovation efforts in the other sectors. This mechanism leads to catching-up of the relative productivity of other sectors over some time. Increase in productivity in one sector is thus followed by increases in productivity in the other sectors and the initial innovation generates persistent movement in aggregate productivity.

1 Introduction

When explaining business cycles, the most often used driving force is exogenous stochastic changes in productivity. Serious effort has been devoted to studying effects of changes in uncertainty about these shocks, their volatility or news about their impact in the future. Much less is known about what these shocks actually are.

In this paper, I build a multi-sector model with endogenous innovative activity which allows me to explain changes in aggregate productivity with individual innovations. One type of shocks hence drives both long term economic growth and fluctuations at business cycle frequencies. This model therefore provides an integrated framework to study both economic phenomena. The model, despite having completely serially uncorrelated shock, can generate about a fifth of volatility at business cycle frequencies.

The global solution of the model is obtained numerically using a projection algorithm. I extend the existing procedures dealing with highly dimensional state-spaces[1] by developing a method which adaptively moves and extends the grid used in the projection. As far as I

[1] I use ergodic grid suggested by Judd et al. (2012).

know, this is the first paper which solves a business cycle model with endogenous growth using a global method.

Changes in productivity are at the core of virtually all modern models of business cycles. The standard strategy is to assume that productivity is an exogenous, highly persistent process which augments labour in the production function. Real business cycle literature successfully showed that in a very simple model this approach can explain a great part of movements in endogenous variables. While several other shocks have also been proposed in order to fit the data we observe in reality, much less is known about the true sources of aggregate fluctuations. Some interpret the term "productivity" literally and argue that the Solow residual fits well with the basic model (Prescott, 1986). However, changes in aggregate demand, monopoly pricing (Hall, 1987) or labour effort can also lead to fluctuations in measured TFP. Moreover, there is research exploring settings in which other shocks, for example wealth shocks or variation in firm entry costs, can generate responses which would look like TFP shocks (Jaimovich, 2007; Huo and Rios-Rull, 2012).

In this paper, changes in productivity of final goods production are direct consequences of innovative activity. In this environment, production firms earn profits because of differentiation of final goods. These profits motivate research firms to invest into research and innovation, hoping to replace the incumbent producer and start earning the profits. The incremental changes in productivity are the result of an endogenous labour intensive innovation process

The only shocks in my model are non-persistent stochastic factors affecting the chances of research firms to succeed in their innovation attempts. Research firms receive noisy signals about the quality of their innovation projects and decide optimally how much extra labour they want to use in the innovation process. This extra labour has two effects: On the one hand, it increases the odds of succeeding in research and hence of replacing the incumbent production firm. On the other hand, it directly increases future profits.

The fruits of research are available only one period after the research project is initiated and hence signals about the quality of research projects today are informative about changes in productivity tomorrow. Thus they take on the role of news shocks. I also show that in a simple setting, the quality of the research project increases the returns to research and so good news generate an increase in labour demand. Finally, complementarity of output of individual industries is able to generate a positive response of research in one industry to positive news in other industries due to increases in expected future demand.

One of the asset pricing puzzles is to explain high volatility of asset prices relative to the underlying stream of profits. In this paper, production firms face a risk of being replaced every period. The innovation effort depends both on the idiosyncratic conditions of the research firm (quality of its project) as well as on aggregate conditions and so does the value of the production firm.

The propagation channel in this paper can explain several aspects of business cycles which were previously thought of as independent. First, it generates the correct co-movement both at the aggregate level and across industries. Second, it is consistent with the empirical evidence on the pro-cyclicality of R&D. And finally, modelling the innovation process explicitly is consistent with the empirical evidence on the importance of so called news shocks (Barsky and Sims, 2011; Schmitt-Grohe and Uribe, 2012).

On a more fundamental level, my paper provides an explanation of what technological and news shocks are and how they relate to other endogenous variables. In my model, the news shock is the change of the research prospects which increases the mean productivity tomorrow. However, this "shock" affects the research production function today, whereas the standard RBC news shock does not affect any supply side variable today. This implies that in the standard RBC setting, the reaction to the shock has to come through the demand side (for example wealth or capital adjustment related) which necessarily limits the transmission mechanism. On the other hand, in my model good news, i.e. good research prospects, result in increased R&D spending on labour which ultimately generates the improvement in productivity.

Finally, the aggregate growth is generated by individual innovations which result in jumps in aggregate productivity. These jumps are only positive so there is no problem with technological regress; de-trended RBC models require that the technological process has to cause the aggregate productivity to decrease at significant percentage of time.[2]

Related literature My paper applies the ideas of endogenous growth literature to explain fluctuations on business cycle frequencies. In particular, the model is a finite number of sector version of quality ladder models (Grossman and Helpman, 1991; Aghion and Howitt, 1992). The finite number of industries is needed in order to generate sampling variation which negates the law of large numbers. Dupor (1999) shows that a large number of sectors due to the law of large numbers necessarily decreases aggregate volatility, while Horvath (1998) shows that the power of this result diminishes if some sectors are more important producers of intermediate goods than others and Acemoglu et al. (2012) characterise the relationship between disaggregation decay and network structure of the economy. In my model, the substitutability of the products of different sectors is given by CES preferences of the household while the capital stock is perfectly transformable from one sector another.

The ideas of endogenous growth have been applied to the investigation of business cycles before (Maliar and Maliar, 2004; Ozlu, 1996; Lambson and Phillips, 2007; Andolfatto and MacDonald, 1998; Wälde, 2005). The closest papers to my setting are Phillips and Wrase (2006) and Gertler et al. (2009). While Phillips and Wrase (2006) ask similar question, their model does not generate enough amplification of sectoral shocks what would generate sizable aggregate fluctuations.[3]

[2]See King and Rebelo (1999) for discussion of this point.

[3]Phillips and Wrase make several simplifications which limit the potential strength of the transmission mechanism. In particular, the household labour decision is the key to the transmission in my model, whereas the labour is supplied inelastically in their model. Second, they model the household preferences over the goods by

If changes of productivity are results of innovation, then variations in resources devoted to R&D should be able to help explain cyclical fluctuations. While this relationship is difficult to measure due to noisiness of R&D data, there is mild consensus that R&D is pro-cyclical (Barlevy, 2007; Comin and Gertler, 2006; Walde and Woitek, 2004).

Jorgenson (1988) summarizes earlier findings and documents that about one fourth of economic growth can be explained by increases in productivity, however, the 80% of decline in the growth rate in 70's was caused by the productivity slowdown. Rotemberg (2003) studies the effects of diffusion of technological innovation. He finds that slow diffusion shocks of new technologies of plausible size can generate smooth paths for GDP. He also argue that correct filtering is crucial to get the correct trend. More recently, Gertler et al. (2009) investigate how technology adoption affect business cycles. They find significant response at medium run frequencies and find significant asset price implications.

In a recent paper, Akcigit et al. (2013) allow the innovations to be patented and sold on the market. The conceptual difference is that in their model, there is a degree of miss-match between the type of innovation and a firm using it, which then motivates the trade. Innovating firms can decide how much to spend on innovation, but these costs increase the chance of being successful in innovating, but not the usefulness of this innovation to the firm is completely random. This implies that if perfectly matched, each innovation would deliver the same productivity improvement to its most fitting user. This is very different to my model, where the ideas are not transferable, but "usefulness" is a stochastic outcome of labour intensive innovation production.

Another strand of literature examined the extent to which R&D drives economic growth and the returns to the investments to R&D. Empirical studies focused on firm level data and industry level.[4] While I assume that it is only the entrant who innovates, in traditional schumepeterian models, it is not clear whether it should be the entrants, or rather the incumbents who want to keep their profits. Aghion et al. (2001) build a formal model and document the link between competitiveness and innovation and Aghion et al. (2005) provide empirical evidence for an U-shaped relationship between competition and innovation. Aghion et al. (2009) document the implication of threat of entry on innovation activity of incumbents; in the industries on the technological frontier the threat stimulates the incumbents to innovate, in the lagging industries not so much.

There is also a large literature on the effects of productivity shocks. The traditional way how to identify productivity shock is to assume that only those can have long run real effects (Blanchard and Quah, 1989) and impose this as a restriction in a VAR model. Consistent with RBC models, increase in productivity should increase marginal product of labour,

Armington aggregator, which, unlike the standard CES which I use, imply that all firms earn the same profits and hence eliminates any role for relative productivity between different sectors. Finally, the research problem in their paper is altogether different, as the research firm cannot react to the different profits in the production sector, and also by allowing only for constant steps in the productivity. These simplifications are convenient to make the model more tractable, however, only at the cost of loosing the internal propagation.

[4] See Griliches (1998) for overview.

increasing wages and decreasing unemployment. However, Galí (1999) found a negative effect on employment in the short run. This finding open big discussion on identification and was confirmed later by some (Francis and Ramey, 2005; Kimball et al., 2006) and questioned by others (Christiano et al., 2004). Leaving identification question aside, the assumption of neutrality of demand shocks in the long run does not hold in endogenous growth models. An alternative identification strategy is to use some measure of changes in productivity in the VAR analysis directly. Unfortunately, probably because there is a significant lag between the innovation itself and its implementation, these indicators do not provide a credible counterpart to TFP (Shea, 1999). To address this delay, Alexopoulos (2011) constructs an index of technological innovation using the publication of manuals as measure of innovation. These publications are expected to appear exactly around the time when an innovation is being introduced into production. She finds positive correlation between this index and TFP, capital investment and labour.

Outline The structure of my paper is the following. I first introduce the model. Then I describe my solution procedure. I calibrate the model and present some preliminary results. Finally, I introduce a possible policy experiment and conclude. Appendix contains details on derivation of model equation and extra results.

2 Model

The basic structure of the model is depicted in figure 1. There are three types of agents. There is one representative household and there are N industries, each with one production firm and one research firm. The production firms produce differentiated output. The research firms are trying to innovate and, if successful, will replace the production firm in its industry.

In my model, future productivity is determined by the innovation success of the research firms. Using labour, these research firms produce some innovation, which can be either successful or not. Importantly, each period each research firm receives an industry specific signal about the quality of its research project.

These signals play the role of news shocks, because they are informative about the expected future productivity. However, unlike standard news shocks, they directly affect the incentives of the research firms today. In particular, research firms decide about optimal labour inputs and a better signal makes a research firm hire more researcher. This is the mechanism how my model tries to overcome the problem of Pigou cycles. In standard RBC models, news about the future higher productivity decrease labour supply due to a positive wealth effect. In my model, the increase of labour demand in the research sector outweighs the wealth effect.

Secondly, my model generates sectoral co-movement. I will show that if consumers view the different products as complements, then innovation in one sector generates positive

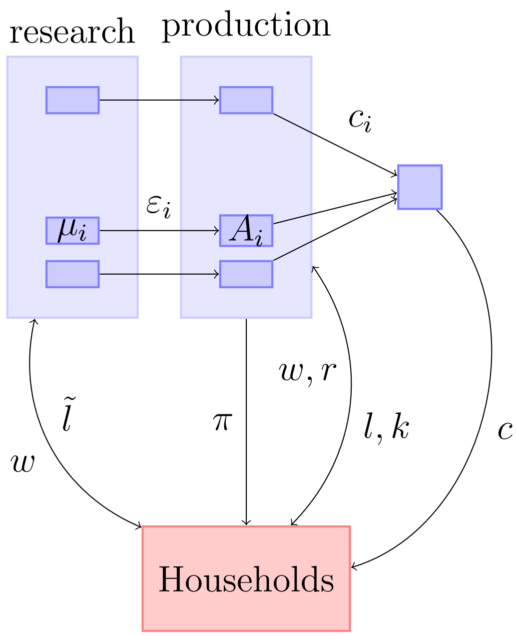


Figure 1: Basic scheme of the model

externalities for produces in all other sectors. The reason is that after increasing production in one sector, due to the complementarity, the household starts to value the other products more which leads to an increase in marginal profits in the remaining industries. This means that the research firms in all the other sectors now face stronger incentives to innovate. However, this also means that already the news about a good research project in one industry increases the incentives to innovate in all other industries. Hence, a purely idiosyncratic shock in one industry has an aggregate effect.

Now I describe the three types of agents in my model in detail.

2.1 Household

The households work, rent out capital, consume, accumulate capital and invest in research and production firms.

2.1.1 Utility function

The household problem is to maximize its discounted expected utility

$$\max_{\{C_t, L_t\}_{t=0}^{\infty}} \mathsf{E}\left[\sum_{t=0}^{\infty} \beta^t u(C_t, l_t, \tilde{l}_t)\right],$$

where $l_t = \frac{1}{N}\sum_{i=1}^{N} l_{it}$ is the total labour supply to the production firms and $\tilde{l}_t = \frac{1}{N}\sum_{i=1}^{N} \tilde{l}_{it}$ is the total labour supply to the research firms. The $1/N$ normalisation keeps the size of the household constant relative to the number of industries.

Household period utility function $u(C, l_t, \tilde{l}_t)$ is assumed to be

$$u(C, l_t, \tilde{l}_t) = \frac{C^{1-\gamma}}{1-\gamma}(1 - (l_t + \tilde{l}_t))^{-\phi}, \quad \gamma > 1, \phi > 0, \tag{1}$$

where $l, \tilde{l} > 0$ and $l + \tilde{l} < 1$. This implies that the "research" labour force is fully substitutable by the "production" labour force. Consumption C denotes the CES consumption aggregate, defined as

$$C_t = \left[\frac{1}{N}\sum_{i=1}^{N} C_{it}^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}}, \tag{2}$$

where N is the number of industries/products. The weighting makes sure that a mere increase in the number of industries does not automatically increase utility and eliminates love for variety. In this paper, the number of industries is exogenous, so this has normalisation has no economic implications. The aggregator implies that the optimal demand for consumption good i, C_i is a fraction of aggregate consumption C given by the relative price $\left(\frac{p_i}{P}\right)^{-\theta}$:

$$C_{it} = \left(\frac{p_{it}}{P_t}\right)^{-\theta} C_t. \tag{3}$$

The details of the derivation are given in appendix A.1, page 33.

2.1.2 Household budget constraint

The household earns wages by working in the production and research firms $W_t \sum_{i=1}^{N}(l_{it} + \tilde{l}_{it})$, capital income $r_t \sum_{i=1}^{N} K_{it}$ earned by out the capital and receives dividends from owning shares of the production firms (the fraction denoted by s_{it}^P). The production firms distribute all their profits, so households obtain $\sum_{i=1}^{N} \Pi_{it} s_{it}^P$. Furthermore, the household can potentially sell the shares of the production firms at price Q_{it}^P and the accumulated capital $(1-\delta)K_t$.

On the expenditures side, the household buys consumption C_t, accumulates capital K_{t+1} and buys shares s_{it+1}^R and s_{it+1}^P of research and production firms at prices Q_{it}^R and Q_{it}^P. One unit of capital K_t can costlessly be transformed into one unit of consumption good C_t.

The households own the production and research firms. When born, research firm issue equity to finance the research labour. However, not 100% of the equity is sold to households. The part which is not sold is held by a mutual fund. While the mutual fund is also ultimately own by the households, this separation means that the research firms receive exactly as much funds they need to finance their labour costs. When distributing profits, production firms pays s^P fraction directly to the households. The rest of profits, denoted by D, is paid out to the mutual fund, which then distributes the money back to the households. This means that the households receives all the profits every period, while the value of assets can be determined in a traditional way.

The budget constraint is (for derivation see appendix A.2 on page 34)

$$C_t + K_{t+1} - (1-\delta)K_t = \frac{r_t}{P_t}K_t + \frac{W_t}{P_t}(l_t + \tilde{l}_t) + \frac{1}{N}\sum_{i=1}^{N} s_{it}^P\left(\frac{\Pi_{it}}{P_t} + \frac{Q_{it}^P}{P_t}\right) - \frac{1}{N}\sum_{i=1}^{N}\left(s_{it+1}^P\frac{Q_{it}^P}{P_t} + s_{it+1}^R\frac{Q_{it}^R}{P_t}\right) + D_t \tag{4}$$

where the household chooses C_t, K_{t+1} and s_{it+1}^P, s_{it+1}^R, taking W_t, r_t, Q_{it}^P and Q_{it}^R as given (for all industries $\forall i = 1, \ldots, N$). D_t represents the rest of the profits of the production and research firm.

Success of a research firm has two implications. First, the incumbent firm is replaced and hence its stocks lose all value. Second, s_{it+1}^R becomes s_{it+1}^P, because the research firm now becomes the production firm and the owners do not change.

2.1.3 Consumer optimality conditions

The lagrangian is

$$\mathscr{L} = \sum_{t=0}^{\infty}\beta^t\Bigg[u(C_t, l_t, \tilde{l}_t) - \lambda_t\left(C_t + K_{t+1} - (1-\delta)K_t + \frac{1}{N}\sum_{i=1}^{N}\left(s_{it+1}^P\frac{Q_{it}^P}{P_t} + s_{it+1}^R\frac{Q_{it}^R}{P_t}\right)\right) + \lambda_t\left(\frac{r_t}{P_t}K_t + \frac{W_t}{P_t}(l_t + \tilde{l}_t) + \frac{1}{N}\sum_{i=1}^{N}s_{it}^P\left(\frac{\Pi_{it}}{P_t} + \frac{Q_{it}^P}{P_t}\right)\right)\Bigg]$$

The first order conditions are (for detailed derivation see see appendix A.3 on page 35)

$$\begin{aligned}
u_{C_t} &= \lambda_t \\
-\frac{u_{l_t}}{\frac{W_t}{P_t}} &= \lambda_t \\
-\frac{u_{\tilde{l}_t}}{W_t} &= \lambda_t \\
\beta\lambda_{t+1}\left(1 - \delta + \frac{r_{t+1}}{P_{t+1}}\right) &= \lambda_t \\
\beta\lambda_{t+1}\frac{P_t}{P_{t+1}}\frac{\bar{\mathbb{1}}_{it}(\Pi_{it+1} + Q_{it+1}^P)}{Q_{it}^P} &= \lambda_t \\
\beta\lambda_{t+1}\frac{P_t}{P_{t+1}}\frac{\mathbb{1}_{it}(\Pi_{it+1} + Q_{it+1}^P)}{Q_{it}^R} &= \lambda_t
\end{aligned}$$

where $\mathbb{1}_{it}$ is an indicator function which captures the success of the research firm i in period t. This research firm becomes the production firm in the period $t+1$. The complement indicator $\bar{\mathbb{1}}_{it}$ captures the case where the research firm in industry i has not been successful so that the incumbent production firm remains the active production firm in period $t+1$.

Combining the conditions I get the labour supply condition:

$$-\frac{u_{C_t}}{P_t} = \frac{u_{l_t}}{W_t}$$

and the Euler equation:

$$u_{C_t} = \beta \mathsf{E}\left[u_{C_{t+1}}\left(\frac{r_{t+1}}{P_{t+1}} + 1 - \delta\right)\right]$$

Let's define the stochastic discount factor as $m_{t+1} = \beta \frac{u_{C_{t+1}}}{u_{C_t}}$, then the Euler equation can be written as

$$1 = \mathsf{E}\left[m_{t+1}\left(\frac{r_{t+1}}{P_{t+1}} + 1 - \delta\right)\right]$$

giving the condition on expected return to saving in the capital stock. This return has to be equal to the alternative saving sources, investing into stocks of research and production firms. For each industry $i = 1, \dots, N$, the expected returns must satisfy

$$1 = \mathsf{E}\mathbb{1}_{it} m_{t+1} \frac{P_t}{P_{t+1}} \frac{\Pi_{it+1} + Q^P_{it+1}}{Q^R_{it}} \tag{5}$$

$$1 = \mathsf{E}\bar{\mathbb{1}}_{it} m_{t+1} \frac{P_t}{P_{t+1}} \frac{\Pi_{it+1} + Q^P_{it+1}}{Q^P_{it}} \tag{6}$$

If the utility function is $u(C, l, \tilde{l}) = \frac{C^{1-\gamma}}{1-\gamma}(1 - \frac{\sum_i l_{it} + \tilde{l}_{it}}{N})^{-\phi}$ and using the convention $l_t = \frac{\sum_i l_{it}}{N}$, then the labour supply condition is

$$C_t = \frac{\gamma - 1}{\phi}(1 - (l + \tilde{l}))\frac{W_t}{P_t}.$$

The stochastic discount factor is

$$m_{t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \left(\frac{1 - (l_{t+1} + \tilde{l}_{t+1})}{1 - (l_t + \tilde{l}_t)}\right)^{-\phi}$$

2.2 Production firm

The problem of the production firm is simple; given the factor prices (wage rate W and interest rate r) it chooses labour and capital to maximize its profits, subject to a limit pricing constraint to drive less productive competitors out of the market. The problem is simple, because the optimisation is static. This is because the capital accumulation is decided by the household. The reason for this is that production firms are replaced by the entering research firms, hence it is easier if the capital is held by the only agent in the model which cannot be replaced.

Production choices The optimal choices of the production firms are found by cost minimisation, taking into account the demand function of its product Y_{it} at its price P_{it}. The production function in each industry is standard Cobb-Douglas

$$Y_{it} = K_{it}^{\alpha}(A_{it}l_{it})^{1-\alpha},$$

where A_{it} is the productivity of the production firm in industry i in period t. The problem is to find

$$\begin{aligned} \min_{l_{it},K_{it}} \quad & W_t l_{it} + r_t K_{it} \\ \text{s.t. } & K_{it}^{\alpha}(A_{it}l_{it})^{1-\alpha} = Y_{it} \end{aligned}$$

where W and r denote nominal wage rate and interest rate. The problem is identical in all industries (apart from differences in A_{it}).

I assume away a strategic behavior affecting the decisions of individual firms, so the firms still take aggregate prices as given when they decide about their optimal price. This allows to solve the model easily and the higher is the number of industries, the less problematic this assumption is.

Given the fact that different production firms in different industries share the same Cobb-Douglas production function (they differ in their productivity A_{it}), the ratio of factors is the same for all of them,

$$\frac{K_{it}}{l_{it}} = \frac{W_t}{r_t}\frac{\alpha}{1-\alpha}, \tag{7}$$

so higher W relative to the r increases K relative to l. The important insight here is that the firm will always use a fixed proportion of capital to labour, $K/l = \frac{W}{r}\frac{\alpha}{1-\alpha}$, no matter the firm specific productivity A_{it} is.

Next, this result is substituted into the production function to get $Y(l)$, which can be inverted to get $l(y)$ and $K(y)$, taking W, r as given. The cost function is $\psi(Y) = Wl(Y) + rK(Y)$, the marginal costs are then $\mathrm{d}\psi/\mathrm{d}Y = W\mathrm{d}l/\mathrm{d}Y + r\mathrm{d}K/\mathrm{d}Y$. Solving this yields

$$MC(Y_{it}) = A_{it}^{\alpha-1}W_t^{1-\alpha}r_t^{\alpha}\alpha^{-\alpha}(1-\alpha)^{\alpha-1}. \tag{8}$$

The marginal costs are scale invariant (with respect to Y), so different firms (in different industries) differ only due to the difference in the productivity A_{it} regardless of their level of production Y_{it}. As expected, marginal costs are rising in W and r and decreasing in firm productivity,

$$l(Y_{it}) = Y_{it}\left(\frac{W_t}{r_t}\frac{\alpha}{1-\alpha}\right)^{-\alpha} A_{it}^{-1+\alpha} \tag{9}$$

$$K(Y_{it}) = Y_{it}\left(\frac{W_t}{r_t}\frac{\alpha}{1-\alpha}\right)^{1-\alpha} A_{it}^{\alpha-1} \tag{10}$$

2.3 Research firm

In my model, each period a new research firm is born in every production sector and tries to innovate upon the existing production technology. If the research firm is successful in innovation, it becomes a production firm in its industry in the following period. It will then charge the limit price and drive the incumbent production firm out of the market. It will therefore be the only production firm in its industry and will gain profits until it is replaced by another successful research firm in the future.

The fact that research firms observe only a noisy signal implies that the final outcome of the research process is a stochastic variable which can be predicted using the initial signal and the research labour used in the innovation process. To make the notation simpler, I transform the research problem such that the signal extraction is avoided; from now I will assume that the firm learn the true quality of the project perfectly, but there is a new "luck" shock which comes after the research labour is used. This way the outcome is still unknown yet predictable random variable when deciding about the labour inputs while the signal extraction is avoided.

The success of a research firm is determined by three factors: by the observed quality of the project $\mu \sim N(\mu_0, \sigma_\mu^2)$, by the amount of labour employed to improve the project $\tilde{l}$ and finally and an idiosyncratic shock $\varepsilon \sim N(0, \sigma_\varepsilon^2)$ that represents luck. The final research output is then $\mu + f(\tilde{l}) + \varepsilon$ where $f(\cdot)$ is the research production function. An innovation project is successful if this research output is positive. If it is negative the innovation failed and the research firm exits the model. Innovation steps are defined as

$$e_{it+1} = \max\{0, f(\tilde{l}_{it}) + \mu_{it} + \varepsilon_{it}\} \tag{11}$$

If the innovation is successful, $e_{it+1} > 0$, the research firm enters the production sector in the next period with a productivity bigger than that of the current production firm by the factor $1 + e_{it+1}$.

For research production function, I use

$$f(\tilde{l}_{it}) = \tilde{\alpha}_0 \tilde{l}_{it}^{\tilde{\alpha}_1}, \tag{12}$$

so the research production function has diminishing returns to scale parametrised by $\tilde{\alpha}_1$ with slope coefficient $\tilde{\alpha}_0$.This is in line with literature which typically assume that innovation are generated using labour only.

Shocks The only two shock in the model are quality of innovation μ and luck factor ε. The two shocks are uncorrelated with each other, across time and industries. I assume that both are distributed normally, $\mu_{it} \sim N(\mu_0, \sigma_\mu^2)$ and $\varepsilon_{it} \sim N(0, \sigma_\varepsilon^2)$.

I parametrise μ_0 to be a negative number. This means that without any labour effort, the chances of innovation are low. Formally, the probability of the innovation being successful can be found by finding the threshold value $\bar{\varepsilon}$, such that $\forall \varepsilon_{it} > \bar{\varepsilon}$, the research satisfies $f(\tilde{l}_{it}) + \mu_{it} + \varepsilon_{it} > 0$. Clearly $\bar{\varepsilon} = -(f(\tilde{l}_{it}) + \mu_{it})$ and using the properties of normal distribution the probability of innovation success can be found to be

$$\mathsf{P}(e_{it+1} > 0 | \mu_{it}) = \Phi\left(\frac{\mu_{it} + f(\tilde{l}_{it})}{\sigma_\varepsilon}\right), \tag{13}$$

where Φ is standard normal cumulative distribution function. Similarly, the unconditional expected innovation step can be found using standard results about truncated normal distribution. The unconditional innovation step will be closely related to aggregate growth rate generated by my model and this fact will be used to calibrate $\mu_0, \sigma_\mu, \sigma_\varepsilon, \tilde{\alpha}_0$ and $\tilde{\alpha}_1$.

Pricing A successful research firm enters the production sector in the following period and sets its price in *limit pricing*. This means that it sets the price so that it drives the incumbent production firm out of the market by charging a price equal to the incumbent's marginal costs of production.

Marginal costs of producing output volume y do not depend of y because of constant returns to scale in both input factors, see equation (8). The difference between the marginal costs of the incumbent and those of the successful research firm is that the latter is more productive by the factor $1+e_{it+1}$, the recent enhancement in productivity. The productivity of the recently successful research firm A_{it} therefore has the form

$$A_{it} = (1 + e_{it})\tilde{A}_{it} \tag{14}$$

where $\tilde{A}_{it}$ is the productivity of the incumbent production firm. The marginal cost of the successful research firm (MC) relative to those of the incumbent firm ($\widetilde{MC}$) hence satisfy

$$\widetilde{MC} = (1 + e)^{1-\alpha} MC$$

Therefore, in order to drive out the incumbent firm, the new entrant can charge a mark-up of $(1 + e)^{1-\alpha}$.

Note that the incumbent firm and the new entrant use the same ratio of K/l (see equation (7)). The incumbent firm just needs to employ *more of both* factors (in the same proportion) to produce the same amount of output.

Hence the price charged for goods in sector i is

$$P_{it} = \frac{1}{\tilde{A}_{it}^{1-\alpha}} W_t^{1-\alpha} r_t^{\alpha} \alpha^{-\alpha} (1-\alpha)^{\alpha-1} \tag{15}$$

This price assumes that the firm uses the optimal choice of inputs. Note that a higher productivity of the incumbent firm $\tilde{A}$ forces the new firm to charge a lower price. Due to the limit pricing, it is *the previous, not the current*, generation of firms whose productivity determines prices.

I assume that the technology progress embodied in enhancements e is small enough that the pricing condition to drive out the incumbent firm is binding so that firms are not charging fully monopolistic prices. The more productive firm cannot charge its monopolistic price, because the incumbent firm could charge a lower price and serve the whole market. This implies that charging the limit price, i.e. charging a price equal to the marginal costs of the incumbent, is the profit maximizing strategy. In this case the incumbent firm cannot realize profits and is driven out of the market. This assumption therefore guarantees that there is always only one firm producing in each market.

Given the CES consumption aggregator, the standard price index has form of

$$P_t = \left[\frac{1}{N} \sum_{i=1}^{N} P_{it}^{1-\theta} \right]^{\frac{1}{1-\theta}} = W_t^{1-\alpha} r_t^{\alpha} \alpha^{-\alpha} (1-\alpha)^{\alpha-1} \left[\frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{\tilde{A}_i^{1-\alpha}} \right)^{1-\theta} \right]^{\frac{1}{1-\theta}} \tag{16}$$

The relative price of intermediate good i can also be found to be

$$\frac{P_{it}}{P_t} = \frac{\frac{1}{(\tilde{A}_{it})^{1-\alpha}}}{\left[\frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{\tilde{A}_{jt}^{1-\alpha}} \right)^{1-\theta} \right]^{\frac{1}{1-\theta}}} \tag{17}$$

The higher is the productivity in a sector relative to the others, the lower must be its relative price. To see the intuition, recall the CES aggregator. A more productive firm finds it profitable to produce more, but consuming more of one type of good decreases its utility. In order to sell more goods, it is necessary to decrease the price. Also note that the interest rate and wage rate cancel out and do not play a role for the relative price. The reason is that they affect both individual price P_i and the aggregate price P in the same way.

Finally, I normalise the aggregate price level $P_t = 1, \forall t = 1, \ldots, \infty$. The reason is that the model only contains real variables. For demand, what really matters is the relative price, which is still well defined.

Profits The profit of a production firm (and of a recently successful research firm) is

$$\Pi_{it} = P_{it} Y_{it} - W_t l_{it} - r_t K_{it}$$

which can be solved to get

$$\Pi_{it} = Y_t \tilde{a}_{it}^{(1-\theta)(\alpha-1)} \left[(1+e_{it})^{1-\alpha} - 1 \right], \tag{18}$$

where $\tilde{a}_{it}$ is a measure of the relative productivity of one sector relative to all other sectors and is defined as

$$\tilde{a}_{it} = \frac{\tilde{A}_{it}}{\left\{ \left[\frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{\tilde{A}_{it}^{1-\alpha}} \right)^{1-\theta} \right]^{\frac{1}{1-\theta}} \right\}^{-\frac{1}{1-\alpha}}} \tag{19}$$

The relation with the relative productivity is crucial to my model. This result shows that as long as $\theta < 1$, the profits are decreasing in own relative productivity $\tilde{a}$. This might seem counter-intuitive at first, but it is the implication of the CES preferences affecting the pricing via the price index. In particular, the values of θ below one mean that the goods of different industries are complements rather than substitutes. A more productive firm finds it profitable to produce more than a less productive firm (in a different industry). However, in order to sell its bigger output, the production firm has to lower its price as the consumers marginal utility declines. However, because of the complementarity, the marginal utility of a good declines faster the stronger the degree of complementarity is. A similar mechanism can be found in Acemoglu and Guerrieri (2008).

This means that the complementarity in the consumer utility function generates externalities for production firms and via the expected profits for research firms as well. In particular, conditional on the same value of innovation step e, *if she could choose*, the researcher would rather be in a low productivity industry rather than the high one. This mechanism balances the model so the relative productivities $\tilde{a}$ have mean reverting behavior.

Equation 18 also reveals that profits are scaled with aggregate output Y and increase with the innovation step e. All else equal, the higher the quality of the research project μ, the bigger is the productivity step e, the higher is the mark-up over marginal costs of the incumbent production firm and hence the higher are the profits a successful research firm will earn.

The quality of innovation is therefore important via two channels: First, for given research effort the likelihood of being successful and hence of getting access to future profits increases with the quality of the project. Second, the better the innovation the higher are these profits. The interplay of these two motives makes the problem significantly non-linear.

Asset prices Households invest in shares of both production and research firms. For the value of the production firm their optimality condition therefore requires

$$Q_{it}^{P} = \mathsf{E}\left[\bar{\mathbb{1}}_{it} m_{t+1} (\Pi_{it+1} + Q_{it+1}^{P}) \right] \tag{20}$$

where Q^P_{it} is the value of the production firm in sector i at time t and $\bar{\mathbb{1}}_{it}$ is an indicator function which is equal to one if the production firm survives, i.e. the current research firm is unsuccessful:

$$\bar{\mathbb{1}}_{it} = \begin{cases} 1 & \text{if} \quad \mu_{it} + f(\tilde{l}_{it}) + \varepsilon_{it} \leq 0 \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

The value of a research firm, on the other hand, has to satisfy

$$Q^R_{it} = \mathsf{E}\left[\mathbb{1}_{it} m_{t+1}(\Pi_{it+1} + Q^P_{it+1})\right] \tag{22}$$

where $\mathbb{1}_{it}$ is an indicator function which is equal to one if the research firm is successful (the complement of $\bar{\mathbb{1}}_{it}$):

$$\mathbb{1}_{it} = \begin{cases} 1 & \text{if} \quad \mu_{it} + f(\tilde{l}_{it}) + \varepsilon_{it} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

If the research firm is successful it will become a production firm. Shareholders of this firm will hence get next period's profits and will still own shares in the firm which will then be a production firm of value Q^P_{it+1}. This is reflected in equation (22).

Problem of the research firm All research firms are fully owned by the households; a research firm initially sells s^R_{it} fraction of its share to household on a market, keeping $1 - s^R_{it}$ to itself. The firm is run by the mutual fund, which is ultimately own by the households. The fact that different households can hold different firms means that the firms compete on the market and maximise the firm value rather than directly maximising household utility (for example by maximising employment). Thus the choice of research labor $\tilde{l}$ which maximises the value of the research firm net of labour costs:

$$\arg\max_{\tilde{l}_{it}} \mathsf{E}\left[\mathbb{1}(\tilde{l}_{it}) m_{t+1}(\Pi_i(\tilde{l}_t) + Q^P_{it+1}(\tilde{l}_{it}))\right] - \tilde{l}_{it} W_t \tag{24}$$

Note that both the likelihood of being successful as well as future profits and the value of the production firm directly depend on the choice of current research labour. The reason is that all these terms are a function of the innovation step e_{it+1} which depends on research labour (see equation 11).

2.4 Equilibrium

The sequence of events and action in the model is as follows:

1. First, at the beginning of period t, in all sectors $i = \{1 \dots N\}$ a research firm is born with a research idea of quality μ_{it}. The vector $\boldsymbol{\mu}_t = (\mu_{1t}, \dots, \mu_{Nt})$ summarises the

quality of current research projects in all industries and is public knowledge.

2. Second, based on $\boldsymbol{\mu}_t$ as well as the other state variables research firms issue stocks and with the proceeds hire workers $\tilde{l}_{it}$ to improve their research ideas. Also, households production labor l_{it} is hired, output is produced, households decide how much to save and they consume. Note that $\boldsymbol{\mu}_t$ also affects the decisions of the housholds, both directly through asset prices and through growth expectations via the stochastic discount factor.

3. Finally, at the end of the period (after all markets are cleared and consumption took place), in each sector there is an idiosyncratic shock ε_{it} to research which determines the success of the research project in each sector.

The state vector $\boldsymbol{\Sigma} = [k, \boldsymbol{e}, \boldsymbol{\mu}, \tilde{\boldsymbol{a}}]$ consists of aggregate capital stock k and mark-ups of currently producing firms $\boldsymbol{e}$, quality of project of current research firms $\boldsymbol{\mu}$ and relative productivity of current producers $\tilde{\boldsymbol{a}}$. The dimension of the state vector is $3N + 1$.

An equilibrium is a sequence $\{l_t, \tilde{l}_t, K_{t+1}, r_t, w_t, s^P_{1t+1}, \ldots, s^P_{Nt+1}, s^R_{1t+1}, \ldots, s^R_{Nt+1}\}_{t=0}^{\infty}$ such that

1. $\{l_t + \tilde{l}_t, K_{t+1}, s^P_{1t+1}, \ldots, s^P_{Nt+1}, s^R_{1t+1}, \ldots, s^R_{Nt+1}\}_{t=0}^{\infty}$ solves the household problem
2. $\{\tilde{l}_{it}\}_{t=0}^{\infty}$ solves the research firm problem
3. $\{l_{it}, K_{it}\}_{t=0}^{\infty}$ solves the production problem of the production firm
4. markets for labour and capital clear

2.5 Normalisation

As the productivity grows, the model economy produces more and more output. While there is no fixed trend (like in a model where aggregate productivity has an exogenous growth rate, for example an AR(1) process with drift), it is still possible to define the aggregate level of technology and then show that the variables scale linearly with this technological index.

Observing the aggregate price equation (16), it is natural to define aggregate productivity $\tilde{A}_t$ as

$$\tilde{A}_t \equiv \left\{ \left[\frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{\tilde{A}_{it}^{1-\alpha}} \right)^{1-\theta} \right]^{\frac{1}{1-\theta}} \right\}^{-\frac{1}{1-\alpha}} \tag{25}$$

It follows that if $\tilde{A}_{it} = \tilde{A} \quad \forall i = 1, \ldots, N$, then $\tilde{A}_t = \tilde{A}$. Along the balanced growth path, $\tilde{A}_{it} = \bar{A}_t$, then $\tilde{A}_t = \bar{A}_t$, so aggregate productivity grows together with the growth in the individual industries. From now on, normalised values of non-stationary variables X are denoted by lower-case letters, such that $x_t = \frac{X_t}{\tilde{A}_t}$.

The normalised *relative incumbent productivity* is defined as

$$\tilde{a}_{it} = \frac{\tilde{A}_{it}}{\tilde{A}_t} \tag{26}$$

then the law of motion for the relative productivities is

$$\tilde{a}_{it+1} = \frac{(1 + e_{it+1})\tilde{a}_{it}}{\left[\frac{1}{N}\sum_{i=1}^{N}\left[(1 + e_{it+1})\tilde{a}_{it}\right]^{(\alpha-1)(1-\theta)}\right]^{\frac{1}{(\alpha-1)(1-\theta)}}} \tag{27}$$

also, it is useful to derive the growth rate of productivity g_{t+1}:

$$g_{t+1} \equiv \frac{\tilde{A}_{t+1}}{\tilde{A}_t} = \frac{\left[\frac{1}{N}\sum_{i=1}^{N}\left[(1 + e_{it+1})\tilde{a}_{it}\right]^{(\alpha-1)(1-\theta)}\right]^{\frac{1}{(\alpha-1)(1-\theta)}}}{\left[\frac{1}{N}\sum_{i=1}^{N}\tilde{a}_{it}^{(\alpha-1)(1-\theta)}\right]^{\frac{1}{(\alpha-1)(1-\theta)}}} \tag{28}$$

With the aggregate price level normalisation, equation (16) becomes

$$1 = W_t^{1-\alpha} r_t^{\alpha} \alpha^{-\alpha}(1-\alpha)^{\alpha-1}\tilde{A}_t^{-(1-\alpha)}$$

which in normalised terms leads to

$$1 = w_t^{1-\alpha} r_t^{\alpha} \alpha^{-\alpha}(1-\alpha)^{\alpha-1} \tag{29}$$

Recall the shared shape of production function among different industries implied that the same optimal labour/capital ratio across all industries (equation (7)). This symmetry allows to construct an aggregate demands for factors of production. Using the aggregation rules, $l_t = 1/N\sum_{i=1}^{N} l_{it}$ and $K_t = \frac{1}{N}\sum_{i=1}^{N} K_{it}$, the aggregate level relation between capital and labour has to satisfy

$$k_t = l_t \frac{w_t}{r_t}\frac{\alpha}{1-\alpha}. \tag{30}$$

The normalised labour supply condition becomes

$$c_t = \frac{\gamma - 1}{\phi}(1 - (l + \tilde{l}))w_t. \tag{31}$$

So far, the effect of the normalisation has been similar to having a fixed trend. However, in my model, the productivity does not fluctuate around some fixed trend, it changes in a step-wise fashion, i.e. it never goes down. This has important implications for normalised capital.

Suppose that yesterday the household chose consumption and saving such that, given the expected growth in production, the *normalised* capital today should be at its steady state level. However, suppose that due to a lucky realisation, the growth rate has actually been higher than what was expected yesterday. Given the higher than expected productivity, the *realised* value of *normalised* capital today is *below* its steady state.[5] The absolute level of capital K saved at the end of one period carries to the next period without any change, the normalised capital k is affected by the realised growth and becomes a random variable.

$$k_{t+1} \begin{cases} < k_{t+1|t} & \text{if} \quad \tilde{A}_{t+1} > \mathsf{E}_t \tilde{A}_{t+1} \\ = k_{t+1|t} & \text{if} \quad \tilde{A}_{t+1} = \mathsf{E}_t \tilde{A}_{t+1} \\ > k_{t+1|t} & \text{if} \quad \tilde{A}_{t+1} < \mathsf{E}_t \tilde{A}_{t+1} \end{cases}$$

Normalised equilibrium conditions The system of equations in normalised terms is as follows[6]:

$$1 = \beta \mathsf{E} \left[\left(g' \frac{c'}{c} \right)^{-\gamma} \left(\frac{1-(l'+\tilde{l}')}{1-(l+\tilde{l})} \right)^{-\phi} (r'+1-\delta) \right] \tag{32}$$

$$w = (1-\alpha) \left(\frac{k}{l} \right)^{\alpha} \tag{33}$$

$$r = \alpha \left(\frac{l}{k} \right)^{1-\alpha} \tag{34}$$

$$y = \frac{(k)^{\alpha} l^{1-\alpha}}{\sum_{i=1}^{N} \left[(1+e_i) \tilde{a}_i^{1-\theta} \right]^{-1+\alpha}} \tag{35}$$

$$c = \frac{\gamma - 1}{\phi} (1-(l+\tilde{l}))w \tag{36}$$

$$g'k' = y + (1-\delta)k - c \tag{37}$$

$$g' = \frac{\tilde{A}'}{\tilde{A}} = \frac{\left[\frac{1}{N} \sum_{i=1}^{N} [(1+e_i')\tilde{a}_i]^{(\alpha-1)(1-\theta)} \right]^{\frac{1}{(\alpha-1)(1-\theta)}}}{\left[\frac{1}{N} \sum_{i=1}^{N} \tilde{a}_i^{(\alpha-1)(1-\theta)} \right]^{\frac{1}{(\alpha-1)(1-\theta)}}} \tag{38}$$

$$\tilde{a}_i' = \frac{(1+e_i')\tilde{a}_i}{\left[\frac{1}{N} \sum_{i=1}^{N} [(1+e_i')\tilde{a}_i]^{(\alpha-1)(1-\theta)} \right]^{\frac{1}{(\alpha-1)(1-\theta)}}} \tag{39}$$

$$e_i' = \max\{0, \mu_i + \tilde{\alpha}_0 \tilde{l}_i^{\tilde{\alpha}_1} + \varepsilon_i\} \tag{40}$$

$$q_i^P = \mathsf{E} \left[\mathbb{1}_i m_{t+1} g_{t+1} (\pi_i' + q_i^{P\prime}) \right] \tag{41}$$

$$q_i^R = \mathsf{E} \left[\mathbb{1}(\tilde{l}_i) m' g' (\pi_i' + q_i^{P\prime}) \right] \tag{42}$$

[5]This is equivalent to a one-off unexpected increase in the level of productivity in the Solow growth model.
[6]For all derivations see appendix A - C

3 Solution method

I adapt the framework developed by Judd et al. (2012) (JMM hereafter) to the needs of my model. JMM adapt the standard projection solution method for high dimensionality problems by identifying the steps where the dimensionality causes the biggest increase in computation costs and substitute these for a better suited methods. In particular, they advocate using monomials instead of product quadratures and evaluating the projection on an ergodic set instead of a grid constructed by Cartesian product. Finally, instead of using a minimising routine which would need to compute the Hessian of a problem, they suggest using a fixed point iteration procedure via Galerkin regression. In this section I describe how this framework can be applied to solve my model.

3.1 Projection algorithm

I solve the model starting with a guess for the policy functions for production labour $l_t = l(\boldsymbol{\Sigma}_t, \boldsymbol{\theta}_l)$ and research labour $\tilde{l}_t = \tilde{l}(\boldsymbol{\Sigma}_t, \boldsymbol{\theta}_{\tilde{l}})$, where

$$\boldsymbol{\Sigma}_t = (k_t, \tilde{a}_{1t}, \ldots, \tilde{a}_{Nt}, \tilde{e}_{1t}, \ldots, \tilde{e}_{Nt}, \mu_{1t}, \ldots, \mu_{Nt})$$

is a vector containing all state variables and $\boldsymbol{\theta}_l$ and $\boldsymbol{\theta}_{\tilde{l}}$ are vectors containing the coefficients parameterising the approximation. I use Hermite polynomials as basis functions.

The expectations operator in the Euler equation is evaluated over the distribution of $2N$ independent shocks $\boldsymbol{\mu}_{it+1}, \boldsymbol{\varepsilon}_t$ using Gauss-Hermite quadrature adapted to efficiently account for the kinks in the functions.

1. update $l(\Sigma, \boldsymbol{\theta}_l)$:
   (i) get the wage rage $w = w(\boldsymbol{\Sigma})$ by combining the condition of the optimal factor input ratio
   (ii) given $w(\boldsymbol{\Sigma})$, $l(\boldsymbol{\Sigma}_t, \boldsymbol{\theta}_l)$ and $\tilde{l}(\boldsymbol{\Sigma}_t, \boldsymbol{\theta}_{\tilde{l}})$, $c(\boldsymbol{\Sigma})$ can be solved from labour supply equation
   (iii) k' follows from the budget constraint
   (iv) multiply the both sides of Euler equation by l to get

$$l = \beta \mathsf{E}\left[\left(g'(\boldsymbol{\Sigma})\frac{c(\boldsymbol{\Sigma}')}{c(\boldsymbol{\Sigma})}\right)^{-\gamma}\left(\frac{1-(l'(\boldsymbol{\Sigma},\boldsymbol{\theta}_l)+\tilde{l}'(\boldsymbol{\Sigma},\boldsymbol{\theta}_{\tilde{l}}))}{1-(l(\boldsymbol{\Sigma},\boldsymbol{\theta}_l)+\tilde{l}(\boldsymbol{\Sigma},\boldsymbol{\theta}_{\tilde{l}}))}\right)^{-\phi}(r(\boldsymbol{\Sigma}')+1-\delta)l(\boldsymbol{\Sigma},\boldsymbol{\theta}_l)\right]$$

   and evaluate the right hand side by quadrature on a grid[7] for $\boldsymbol{\Sigma}$.
   (v) use these values as a dependent variable and obtain θ_l^{update} by linear projection
   (vi) apply dampening $\boldsymbol{\theta}_l^{new} = \lambda \boldsymbol{\theta}_l^{update} + (1-\lambda)\boldsymbol{\theta}_l$ to ensure convergence

[7] The grid is constructed following Judd et al. (2012): start with a simulated data from the model, cluster the points and then choose a subset such that every point the the simulated data is closer than some δ.

(vii) update the policy function: $l(\mathbf{\Sigma}, \boldsymbol{\theta}_l) = l(\mathbf{\Sigma}, \boldsymbol{\theta}_l^{new})$

2. update $\tilde{l}(\Sigma, \boldsymbol{\theta}_{\tilde{l}})$:

   (i) given $l(\Sigma, \boldsymbol{\theta}_l)$, iterate $\hat{q}^P(\boldsymbol{\theta}_l)$

$$q_i^P(\Sigma) = \mathsf{E}\left[\bar{\mathbb{1}}_i(\Sigma)m(\Sigma')g(\Sigma')\left\{\pi_i(\Sigma') + q_i^P(\Sigma')\right\}\right]$$

   (ii) • given $\hat{q}^P(\boldsymbol{\theta}_l)$, solve for $\tilde{l}$ which maximise the value generated in each sector:

$$\tilde{l} = \arg\max \mathsf{E}\left[\mathbb{1}_i(\Sigma)m(\Sigma')g(\Sigma')\left\{\pi_i(\Sigma') + q_i^P(\Sigma')\right\}\right] - \tilde{l}(\Sigma)w(\Sigma)$$

   note that $\tilde{l}$ affects the expectations of tomorrows q^P directly by affecting corresponding $\tilde{e}_i$ in the state vector

   • use $\tilde{l}$ to update coefficiens $\boldsymbol{\theta}_{\tilde{l}}$ a policy rule $\tilde{l}(\Sigma, \boldsymbol{\theta}_{\tilde{l}})$

3. repeat steps 1 and 2 with dampening factors until $\boldsymbol{\theta}_l$ and $\boldsymbol{\theta}_{\tilde{l}}$ converge

3.2 Construction of grids

(to be added)

3.3 Generalised impulse-response function

Due to the non-linearity of the model standard impulse response functions (IRF) cannot be employed here for two reasons. First, the virtue of a global solution is that it allows the model to react to the shame shock differently, based on the area of the state space. Second, The fact that I explicitly model the changes in productivity as well as the normalised variables means that I can reconstruct the level variables. Hence I report generalized impulse response functions (GIRFs), which are constructed as follows:

1. Simulate M series of shocks $(\boldsymbol{\mu}_t, \boldsymbol{\varepsilon}_t)$ of length T and save them into matrix $\mathbf{\Omega}^{control}$. This matrix has dimensions $(2N \times T \times M)$ and an element $\mathbf{\Omega}^{control}(k, t)$ is the value of k-th shock[8] at period t in all simulations $1, \ldots, M$. The fact that all the policy rules are symmetric across the sectors means that it does not really matter which industry is hit with the shock, so without any loss of generality I always hit sector one.

2. construct $\mathbf{\Omega}^{test,\mu}$ by adding an extra value of shock ξ to the shock μ at time l in industry 1 :

$$\mathbf{\Omega}^{test,\mu}(k,t,m) = \begin{cases} \mathbf{\Omega}^{control}(k,t,m) + \xi & \text{for} \quad l = t, k = 1 \\ \mathbf{\Omega}^{control}(k,t,m) & \text{otherwise} \end{cases} \tag{43}$$

[8] the ordering of the shocks is the following: at position j=$1, \ldots, N$, there is μ_j and then for j=$N+1, \ldots, 2N$ there is ε_j

and $\mathbf{\Omega}^{test,\varepsilon}$ by adding extra value of shock ξ to the shock ε at time t:

$$\mathbf{\Omega}^{test,\varepsilon}(k,t,m) = \begin{cases} \mathbf{\Omega}^{control}(k,t,m) + \xi & \text{for} \quad l = t, k = N+1 \\ \mathbf{\Omega}^{control}(k,t,m) & \text{otherwise} \end{cases} \tag{44}$$

The burn-in period l is important, because the non-linearity of the model means that the effect of a given shock are different for a different position in the state space. The burn-in period thus allows the model to reach different areas before measuring the response to an impulse.

3. simulate the model for the series of shocks $\mathbf{\Omega}^{test,\mu}(t,m)$, $\mathbf{\Omega}^{test,\varepsilon}(t,m)$ and $\mathbf{\Omega}^{control}(t,m)$ and obtain corresponding values of endogenous normalised variables $\boldsymbol{x}^{test,\mu}(t,m)$, $\boldsymbol{x}^{test,\varepsilon}(t,m)$ and $\boldsymbol{x}^{control}(t,m)$
4. compute levels where needed (capital, output, wages,...) and keep the original variables where appropriate (interest rate, l, $\tilde{l}$) $\boldsymbol{X}^{test,\mu}(t,m)$, $\boldsymbol{X}^{test,\varepsilon}(t,m)$ and $\boldsymbol{X}^{control}(t,m)$
5. percentage generalised impulse response function is then generated by

$$\begin{aligned} \text{girf}^{\mu}(t,m) &= \frac{\boldsymbol{X}^{test,\mu}(t,m) - \boldsymbol{X}^{control}(t,m)}{\boldsymbol{X}^{control}}(t,m) \\ \text{girf}^{\varepsilon}(t,m) &= \frac{\boldsymbol{X}^{test,\varepsilon}(t,m) - \boldsymbol{X}^{control}(t)}{\boldsymbol{X}^{control}(t,m)} \end{aligned}$$

6. Sort the result by m dimension and report percentiles to capture the nonlinearity of the model

4 Results (preliminary and incomplete)

4.1 Calibration

My model generates a richer set of prediction than a standard RBC model. The fact that the innovation is modeled explicitly means that the growth rate is determined endogenously. I am taking the values for parameters with a direct RBC counterpart from the literature. The parameters I am calibrating are $\theta, \tilde{\alpha}_0, \tilde{\alpha}_1, \mu_0, \sigma_\mu, \sigma_\varepsilon$

description	label	value
number of sectors	N	2
production function $y = k^{\alpha} l_p^{-\alpha}$	α	0.34
risk aversion	γ	2
labour disutility	ϕ	1.2
capital depreciation	δ	0.015
discount rate	β	0.99

Table 1: Model parameters taken from the literature

description	label	value
goods complementarity	θ	0.75
research production $\tilde{\alpha}_0 \tilde{l}^{\tilde{\alpha}_1}$	$\tilde{\alpha}_1$	0.64
	$\tilde{\alpha}_0$	1
new innovation mean quality	μ_0	-0.35
sd of innovation idea	σ_μ	0.04
sd of luck shock	σ_ε	0.04

Table 2: Model parameters for calibration excercise

Targeted moments include growth rate of GDP, level of employment as well as the correlations among the endogenous variables at the business cycle frequencies. Table details the calibrated parameter values and table 3 reports the resulting simulated moments and compares them to the corresponding data moments.

I do not aim to match the levels of employment perfectly, after all the model presented in this paper is basically still a RBC model and does not contain any friction (like matching) needed for better fit of unemployment level and dynamics. However, I still require my model to generate a reasonable hours worked/leisure split. Table 3 contains both lower bound, represented by 40 hours work week,and upper bound which does not count 8 hours of sleep per day into leisure.

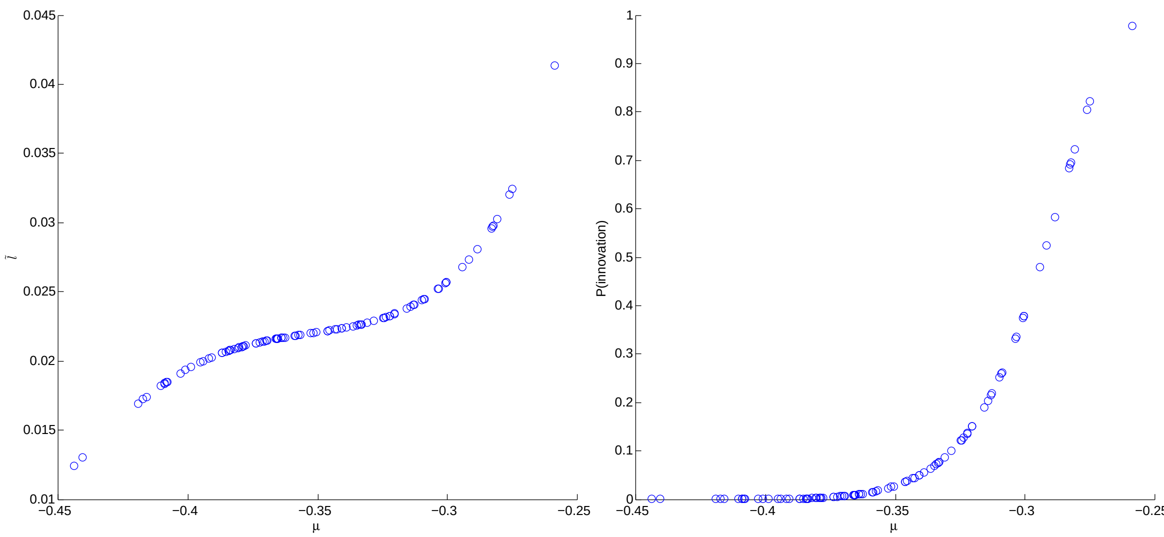
	US data	model
growth rate (y-y)	0.028	0.027
time spent working		0.38
time spent researching		0.03
total hours (40 hours/week)	0.24	
excluding 8 hours of sleep	0.57	

Table 3: Level predictions. Source: FRED, 1947Q1-2013Q4

4.2 Research outcomes

The driving force in my model is the endogenous innovation process. Because the policy functions are nonlinear higher order polynomials, the easiest way how to visualize them is to plot the implied behavior as a function of some interesting state variables.

From figure 2 we see that in the current calibration, it is optimal to spend more work on better projects (panel (a)). Better projects combined with more research labour also are more likely to be successful (panel (b)). Note that for the average quality of research projects, the probability of succeeding is less than 10%. This is an intended result of the parameterisation which tries to capture the fact that most ideas do not make it into production and lots of start-ups never generate any profits.



(a) optimal amount of research labour

(b) likelihood of innovation

Figure 2: Labour in research sector as function of quality of idea shock μ

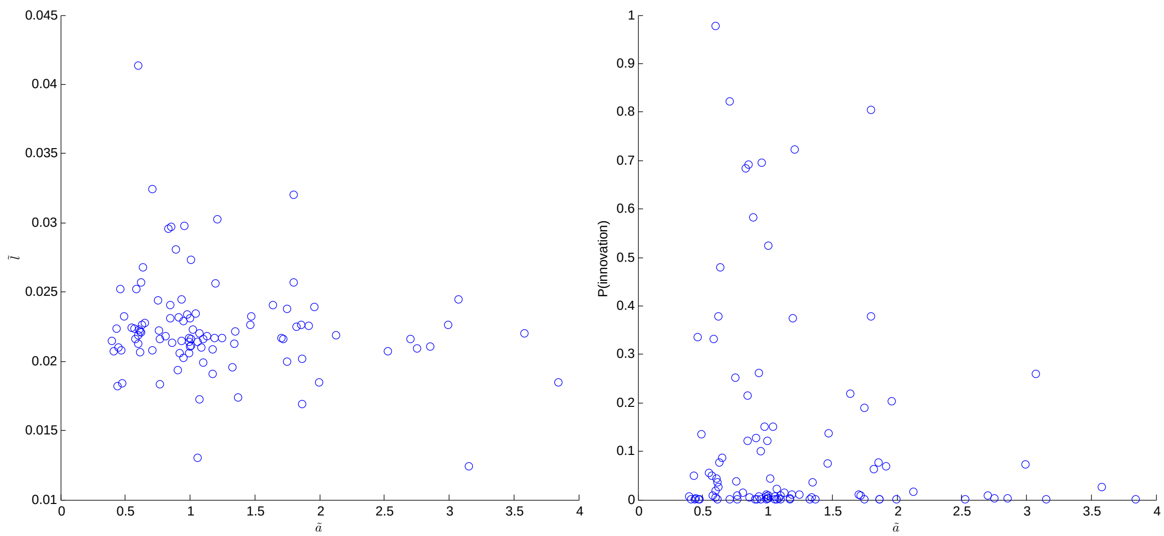
Turning to relative productivity (figure 3), more research labour is applied in the sector which is relatively less productive due to the complementarity of production goods; more productive sectors produce more and increase a value of other goods to consumers. The relatively less productive firms hence enjoy a positive externality from increased demand. Therefore, given the same mark-up, it is more profitable to be a producer in a relatively less productive sector.

However, the research firms ultimately care about the value of the production firm which they might become and profits is only one consideration. The likelihood of being replaced also matters, because it determined the expected number of periods when profits are earned. The extra value attracts more researchers which drives it down. In general equilibrium its not ex ante clear which effect should dominate. However, for the model to be stationary, it is necessary that more research is being done in relative less productive industries. This means that the parameter θ which governs the degree of complementarity is crucial to make sure that the the research concentrates to the less productive industries. It remains to be determined whether this equilibrating mechanism can be strong enough so the model is stationary even for situations when the good are substitutes ($\theta > 1$).

4.3 Level statistics

4.3.1 Stock market capitalisation

In the model, I evaluate the stock market capitalisation as $\sum_{i=1}^{N} q_{it}^{P} + q_{it}^{R}$ averaged over time.



(a) Relative productivity

(b) Relative productivity

Figure 3: Probability of successful innovation

The US stock market capitalisation[9] has very strong low frequency movements. In contrast, the series simulated by the model feature movements at much higher frequency. Furthermore, the simulated data do not contain any trend because the firm values q^P and q^R grow at the same rate as output. However, in the US data there was a clear upward trend in the nineties which ended with the dot-com bubble crash and was declining afterwards.

4.4 Business cycle statistics

4.4.1 Impulse response functions

Generalised impulse response functions are computed using the algorithm described in section 3.3. I simulate 1600 time paths for shocks and then add an extra shock of 1.5 standard deviations to the quality of innovation in period 12.[10]

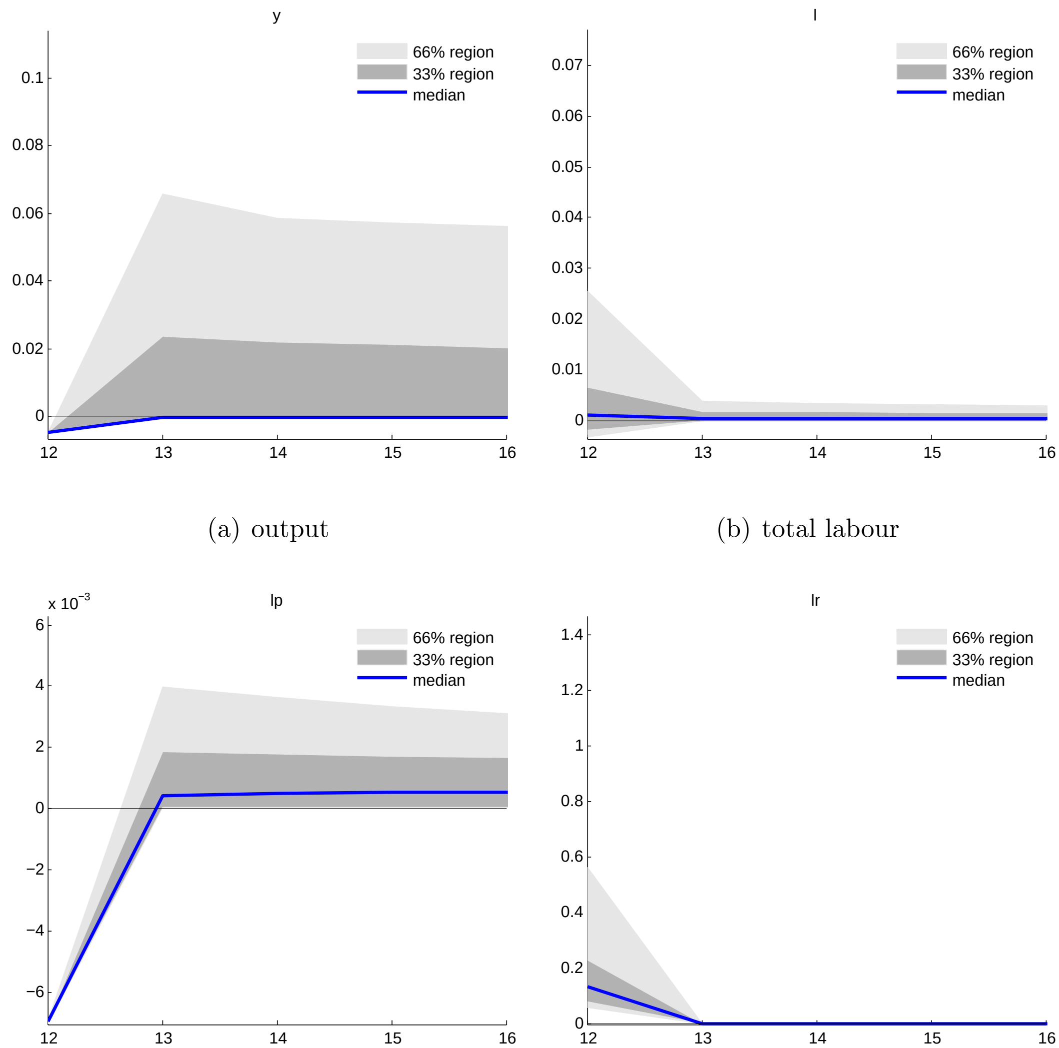
Figure 4 shows the response of the *level* of output (panel (a)), total labour (panel (b)), labour employed in production firms (panel (c)) and labour employed in research firms (panel (d)). We see that initially output falls as production labour is channeled into the research in the particular sector with the good innovation shock. After the initial period, more people are employed in the production sector as the economy needs to accumulate capital to catch up with new level of productivity.

Note also that the standard result in the news shock literature that hours worked fall upon receiving good news about future productivity is not present in this model (measured

[9] Yearly data from FRED, sample period 1989-2011, http://research.stlouisfed.org/fred2/series/DDDM01USA156NWDB

[10] This section contains 3 selected impulse response functions. Other impulse response functions could be found in appendix D.

by the median response). While labour in the production sectors falls, this fall is more than compensated by the increased efforts in the research firms.



(a) output

(b) total labour

(c) labour in production sector

(d) labour in research sector

Figure 4: Selected impulse response functions, shock to the quality of project of the first industry in period 6 ($\mu_{1,6}$)

The nonlinearity is demonstrating itself in the response of the research labour, panel (d). In a linear model, the same shock would always lead to the same response no matter what the state of the world is. In this model, however, the response to the μ can be significantly muted, for example in a situation where the positive shock happens in a sector which is already very productive relative to other industries.

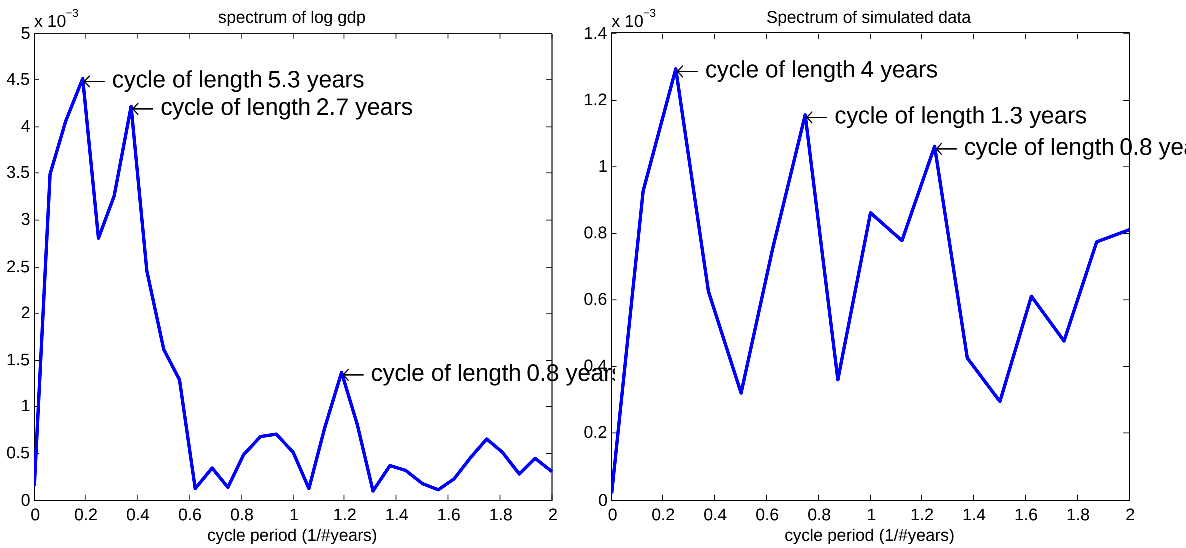
As was noted before, the chances of successful innovation are relatively low even for moderately favorable realisation of the innovation shock μ, so the median difference between the time series with the extra shock and without it is zero. However, the captured non-linearity of the model allows to see the full distribution of possible paths and hence see the positive impact.

4.4.2 Correlations and autocorrelations

Table 4 on page 46 summarizes the correlations and autocorrelations in the simulated data.

4.4.3 Spectral analysis

Spectral analysis can provide valuable insights into the cyclical properties of time series which are not immediately apparent by analysing only auto-correlations in the time domain. Figure 5 displays the estimates of the spectrum of real GDP and of simulated output obtained by averaging the periodogram obtained by Fast Fourier Transformation.



(a) read GDP   (b) simulated output

Figure 5: Spectrum of HP filtered series.

The volatility of the simulated is smaller than the real data. The real GDP spectrum clearly identifies lots of cycles of length from 2-6 years. In the simulated data, There is a peak in the spectrum at frequency corresponding to 4 years. Given that my model was cast in quarterly frequency and the shock in the model are serially uncorrelated, this meas that the internal propagation of the shocks has to be strong to generate cycles of average length of 4 years.

The overall volatility generated by the model is lower than in the data, which can be seen by looking at the scale of y-axis. Looking at the frequencies 2-10 years, we can see that the model can explain roughly 20% of observed variation.

4.5 Estimated TFP vs true changes in productivity

I estimate TFP by assuming an aggregate production function $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$ where K is aggregate capital and L is total labour, which in my model corresponds to $l_t + \tilde{l}_t$. The estimated (logarithm of) TFP is then

$$\widehat{TFP}_t = \frac{\log(Y_t) - \alpha \log(K_t) - (1-\alpha)\log(l_t + \tilde{l}_t)}{1-\alpha}$$

and the true productivity $\tilde{A}$ is defined in (25) on page 16.

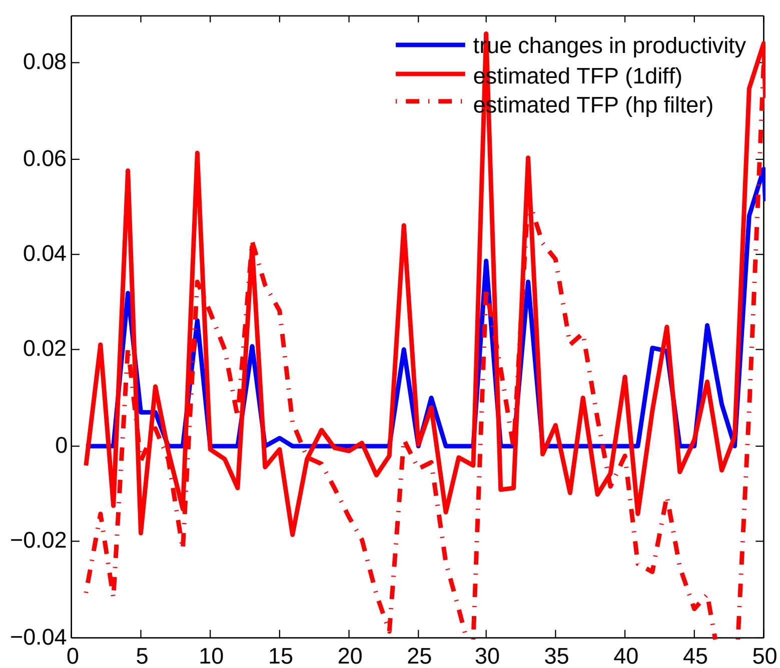


Figure 6: Estimated TFP vs true productivity, first differences of level data.

The comparison between changes in estimated TFP and the true productivity can be seen in figure 6. First, one can immediately see the that true changes in productivity are only positive, yet the estimated TFP shows periods of technological regress. This is regardless of the filtering method. Arguably, the smoothing parameter $\lambda = 1600$ in the HP filter is not optimal for this series as it less persistent than GDP which it was calibrated to.

Nevertheless, any filtering method which is expects to have about the same number of observations below and above the trend will inevitably fail when confronted with series where the changes happen stepwise and the steps are only in one direction.

The knowledge of the true underlying process is hence important for choosing the correct filtering method. Having said that, the correlation between the true and estimated productivities is still high (0.85 in the current simulation).

5 Policy experiment

(to be added)

Research firms in the model maximize the value generated by their research project. However, innovating has a positive externality on other sectors, via increased demand for their products. This externality is not reflected in the decision process of individual research firms and hence there is not enough research being done on the aggregate level. A government could address this inefficiency by subsidizing research labour and by financing this expenditure by imposing a lump sum tax on households.

The expected results are a higher rate of innovation and hence an increased growth rate.

6 Conclusion

One of the fundamental questions of macroeconomics is what causes business cycles. In this paper, I build an integrated framework where both economic growth and business cycle fluctuations are driven by the same shocks. I do so by introducing research firms into a multi-sector RBC model and show how aggregate productivity depends on the results of innovation process in each sector.

In the model, a successful innovation decreases the marginal costs of production in a given industry. In a strict sense this means that the innovation my model describe changes in organization of production rather than introduction of new product varieties. However, endogenous growth literature has described the close relationship between models with fixed productivity but increasing number of varieties with models with fixed number of varieties but growing productivity and hence when drawing conclusions from my model I would like to think about broader definition of research. Research expenditures also should be thought of in a broader sense. For example, Barlevy (2007) notes that up to 40% of research wage bill goes to support staff.

While this paper is still work in progress, I can already show that this setting is capable of matching the aggregate growth while requiring reasonable employment numbers. Furthermore, distinguishing between labour in production firms and research firms allows me to overturn the classical problem in the news shocks literature, i.e. that positive news shocks cause fall in employment due to a wealth effect and the model generates Pigou's cycles at least for total employment and consumption, if not output. Also, as demonstrated by the spectrum of simulated output data, the model already generates persistent cycles.

I also show how estimates of TFP can manifest so called technological regress, i.e. periods of negative productivity growth, while the true productivity process is always bounded by zero.

Regarding the methodology, this paper contributes to the literature by showing how to solve a model where the balanced growth path is affected by optimisation of agents; I partition the model by separating the decisions of agents regarding the production of goods and innovation and develop an iterative algorithm based on projection method to obtain mutually consistent policy rules. Finally, I develop a method how to use Gaussian quadrature to evaluate expectation of variables with kinks. In my setting the kinks are introduced by the fact that research can only generate improvements in productivity as innovations with negative effects are simply not introduced.

Having said that, it is necessary to admit that this is still work in progress. In particular, a model with more sectors and higher order polynomial approximation of the policy function is being solved now. More work will also be done on comparing the model predictions to real data.

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A Households

A.1 Optimal allocation of Consumption Expenditures

CHANGE THE NOTATION: $c_i \to C_i, \quad p_i \to P_i$

Let's assume that the consumption goods are complements with elasticity θ. The household problem of maximizing utility can be solved using duality approach by minimizing expenditures given some budget Z:

$$\max \left[\frac{1}{N} \sum_{i=1}^{N} c_i^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}}$$
$$\text{s.t.} \sum_{i=1}^{N} p_i c_i = Z$$

This constraint problem can be solved using Lagrangian:

$$\mathscr{L} = \left(\frac{1}{N} \sum_{i=1}^{N} c_i^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} - \lambda \left(\sum_{i=1}^{N} p_i c_i - Z \right)$$

taking FOC

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial c_j} &= \frac{\theta}{1-\theta} \left(\frac{1}{N} \sum_{i=1}^{N} c_i^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}-1} \left(\frac{1}{N} \frac{\theta-1}{\theta} c_j^{\frac{\theta-1}{\theta}-1} \right) - \lambda p_j \\ &= \left(\sum_{i=1}^{N} c_i^{\frac{\theta-1}{\theta}} \right)^{\frac{1}{\theta-1}} \left(\frac{1}{N} \right)^{\frac{\theta}{\theta-1}} c_j^{-\frac{1}{\theta}} - \lambda p_j \end{aligned}$$

and hence

$$\lambda = \left(\sum_{i=1}^{N} c_i^{\frac{\theta-1}{\theta}} \right)^{\frac{1}{1-\theta}} \frac{c_j^{-\frac{1}{\theta}}}{p_j} \left(\frac{1}{N} \right)^{\frac{\theta}{\theta-1}}$$

combining this FOC for two different goods, I get

$$\frac{c_j^{-\frac{1}{\theta}}}{p_j} = \frac{c_i^{-\frac{1}{\theta}}}{p_i}$$

which can be rearranged into

$$c_i = c_j \left(\frac{p_i}{p_j} \right)^{-\theta} \tag{45}$$

Now, using the budget constraint $\sum_{i=1}^{N} p_i c_i = Z$, I get

$$Z = \sum_{i=1}^{N} p_i c_j \left(\frac{p_i}{p_j}\right)^{-\theta} = c_j p_j^{\theta} \sum_{i=1}^{N} p_i^{1-\theta}$$

this solved for c_j gives

$$c_j = \frac{Z}{\sum_{i=1}^{N} p_i^{1-\theta}} p_j^{-\theta} = \frac{\frac{1}{N} Z}{\left(\frac{1}{N}\sum_{i=1}^{N} p_i^{1-\theta}\right)^{\frac{1-\theta}{1-\theta}}} p_j^{-\theta} = \frac{\frac{1}{N} Z}{\left(\frac{1}{N}\sum_{i=1}^{N} p_i^{1-\theta}\right)^{\frac{1}{1-\theta}}} \frac{p_j^{-\theta}}{\left(\frac{1}{N}\sum_{i=1}^{N} p_i^{1-\theta}\right)^{\frac{-\theta}{1-\theta}}}$$

and because the price index has form of $P = \left(\frac{1}{N}\sum_{i=1}^{N} p_i^{1-\theta}\right)^{\frac{1}{1-\theta}}$, the last result can be written as

$$c_i = \left(\frac{p_i}{P}\right)^{-\theta} \frac{Z}{P} \frac{1}{N} \tag{46}$$

$$C = \left(\frac{1}{N}\sum_{i=1}^{N} c_i^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}} = \left(\frac{1}{N}\sum_{i=1}^{N} \left[\left(\frac{p_i}{P}\right)^{-\theta} \frac{Z}{P}\frac{1}{N}\right]^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}$$

$$= \frac{Z}{P}\frac{1}{P^{-\theta}}\frac{1}{N}\left(\frac{1}{N}\sum_{i=1}^{N} p_i^{1-\theta}\right)^{\frac{\theta}{\theta-1}} = \frac{Z}{P}\frac{1}{P^{-\theta}}\frac{1}{N}\left(\frac{1}{N}\sum_{i=1}^{N} p_i^{1-\theta}\right)^{\frac{-\theta}{1-\theta}} = \frac{Z}{P}\frac{1}{P^{-\theta}}\frac{1}{N}P^{-\theta} = \frac{Z}{P}\frac{1}{N}$$

and hence

$$PC = \frac{Z}{N}. \tag{47}$$

Combining this results with (46) I get the relative demand of particular good c_i given the aggregate consumption C and relative prices $\frac{p_i}{P}$:

$$c_i = \left(\frac{p_i}{P}\right)^{-\theta} C \tag{46}$$

A.2 Household budget constraint

The budget constraint is (consistent with $PC = Z/N$, see (47))

$$\begin{aligned} & P_t(C_t + K_{t+1} - (1-\delta)K_t) \\ = \frac{1}{N}\left[r_t \sum_{i=1}^{N} K_{it} + W_t \sum_{i=1}^{N} (l_{it} + \tilde{l}_{it}) + \sum_{i=1}^{N} s_{it}^{P}(\Pi_{it} + Q_{it}^{P}) - \sum_{i=1}^{N} \left(s_{it+1}^{P} Q_{it}^{P} + s_{it+1}^{R} Q_{it}^{R}\right)\right] \end{aligned} \tag{48}$$

which can be re-written as

$$\begin{aligned}
C_t + K_{t+1} - (1-\delta)K_t &= \frac{r_t}{P_t}\frac{1}{N}\sum_{i=1}^{N} K_{it} + \frac{W_t}{P_t}\frac{1}{N}\sum_{i=1}^{N}(l_{it} + \tilde{l}_{it}) \\
&\quad + \frac{1}{N}\sum_{i=1}^{N} s_{it}^P\left(\frac{\Pi_{it}}{P_t} + \frac{Q_{it}^P}{P_t}\right) - \frac{1}{N}\sum_{i=1}^{N}\left(s_{it+1}^P\frac{Q_{it}^P}{P_t} + s_{it+1}^R\frac{Q_{it}^R}{P_t}\right) \\
&= \frac{r_t}{P_t}K_t + \frac{W_t}{P_t}(l_t + \tilde{l}_t) \\
&\quad + \frac{1}{N}\sum_{i=1}^{N} s_{it}^P\left(\frac{\Pi_{it}}{P_t} + \frac{Q_{it}^P}{P_t}\right) - \frac{1}{N}\sum_{i=1}^{N}\left(s_{it+1}^P\frac{Q_{it}^P}{P_t} + s_{it+1}^R\frac{Q_{it}^R}{P_t}\right)
\end{aligned}$$

A.3 Consumer optimality conditions

The Lagrangian is

$$\begin{aligned}
\mathscr{L} = \sum_{t=0}^{\infty}\beta^t \Bigg[& u(C_t, l_t, \tilde{l}_t) - \lambda_t\left(C_t + K_{t+1} - (1-\delta)K_t + \frac{1}{N}\sum_{i=1}^{N}\left(s_{it+1}^P\frac{Q_{it}^P}{P_t} + s_{it+1}^R\frac{Q_{it}^R}{P_t}\right)\right) \\
& + \lambda_t\left(\frac{r_t}{P_t}K_t + \frac{W_t}{P_t}(l_t + \tilde{l}_t) + \frac{1}{N}\sum_{i=1}^{N} s_{it}^P\left(\frac{\Pi_{it}}{P_t} + \frac{Q_{it}^P}{P_t}\right)\right)\Bigg]
\end{aligned}$$

The first order conditions are (note the different treatment of l_t and $\tilde{l}_{it}$, the former is the aggregate labour whereas the latter is one industry only)

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial C_t} &= \beta^t\left[u_{C_t} - \lambda_t\right] = 0 \\
\Rightarrow u_{C_t} &= \lambda_t \qquad (49) \\
\frac{\partial \mathscr{L}}{\partial l_t} &= \beta^t\left[u_{l_t} + \lambda_t W_t\right] = 0 \\
\Rightarrow \frac{u_{l_t}}{W_t} &= -\lambda \qquad (50) \\
\frac{\partial \mathscr{L}}{\partial \tilde{l}} &= \beta^t\left[u_{\tilde{l}_t} + \lambda_t W_t\right] = 0 \\
\Rightarrow \frac{u_{\tilde{l}_t}}{W_t} &= -\lambda_t \qquad (51) \\
\frac{\partial \mathscr{L}}{\partial K_{t+1}} &= -\beta^t\lambda_t + \beta^{t+1}\lambda_{t+1}\left(1 - \delta + \frac{r_{t+1}}{P_{t+1}}\right) = 0 \\
\Rightarrow \beta\lambda_{t+1}\left(1 - \delta + \frac{r_{t+1}}{P_{t+1}}\right) &= \lambda_t \qquad (52) \\
\frac{\partial \mathscr{L}}{\partial s_{it+1}^P} &= -\frac{1}{N}\beta^t\lambda_t\frac{Q_{it}^P}{P_t} + \frac{1}{N}\beta^{t+1}\lambda_{t+1}\bar{\mathbb{1}}_{it}\frac{\Pi_{it+1} + Q_{it+1}^P}{P_{t+1}} \\
\Rightarrow \beta\lambda_{t+1}\frac{P_t}{P_{t+1}}\frac{\bar{\mathbb{1}}_{it}(\Pi_{it+1} + Q_{it+1}^P)}{Q_{it}^P} &= \lambda_t \qquad (53) \\
\frac{\partial \mathscr{L}}{\partial s_{it+1}^R} &= -\frac{1}{N}\beta^t\lambda_t\frac{Q_{it}^R}{P_t} + \frac{1}{N}\beta^{t+1}\lambda_{t+1}\mathbb{1}_{it}\frac{\Pi_{it+1} + Q_{it+1}^P}{P_{t+1}}
\end{aligned}$$

$$\Rightarrow \beta \lambda_{t+1} \frac{P_t}{P_{t+1}} \frac{\mathbb{1}_{it}(\Pi_{it+1} + Q^P_{it+1})}{Q^R_{it}} = \lambda_t \tag{54}$$

where $\mathbb{1}_{it}$ is a indicator function which captures the success of the research firm i in the period t. This research firm becomes a production firm in the period $t+1$. The complement indicator $\bar{\mathbb{1}}_{it}$ captures the cases where there has not been success in innovation.

B Firms

B.1 Deriving factor demands by firms

B.1.1 Marginal costs

Given w, r, y, the problem is to find

$$\begin{aligned} &\min_{l,k} \quad wl + rk \\ &\text{s.t. } k^\alpha (e^z A l)^{1-\alpha} = y \end{aligned}$$

Lagrangian

$$\begin{aligned} \mathscr{L} &= wl + rk + \lambda \left(y - k^\alpha (e^z A l)^{1-\alpha} \right) \\ \frac{\partial \mathscr{L}}{\partial l} &= w - \lambda (1-\alpha) k^\alpha (e^z A l)^{-\alpha} e^z A \\ w &= \lambda (1-\alpha) \frac{y}{l} \qquad (55) \\ \frac{\partial \mathscr{L}}{\partial k} &= r - \lambda \alpha k^{\alpha-1} (e^z A l)^{1-\alpha} \\ r &= \lambda \alpha \frac{y}{k} \qquad (56) \\ \frac{\partial \mathscr{L}}{\partial \lambda} &= y - k^\alpha (e^z A l)^{1-\alpha} \\ y &= k^\alpha (e^z A l)^{1-\alpha} \end{aligned}$$

Combining (55) and (56) yields

$$\begin{aligned} w \frac{l}{y} \frac{1}{1-\alpha} &= r \frac{k}{y} \frac{1}{\alpha} \\ k &= l \frac{w}{r} \frac{\alpha}{1-\alpha}, \qquad ((30), \text{ page } 17) \end{aligned}$$

Use this result in the production function

$$\begin{aligned}
y &= k^{\alpha}(e^{z}Al)^{1-\alpha} \\
&= \left(l\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^{\alpha}(e^{z}Al)^{1-\alpha} \\
&= l\left(\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^{\alpha}(e^{z}A)^{1-\alpha} \\
l(y) &= y\left(\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^{-\alpha}(e^{z}A)^{-1+\alpha} && (9) \\
l'(y) &= \left(\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^{-\alpha}(e^{z}A)^{-1+\alpha}
\end{aligned}$$

so if y goes up by unit, the optimal choice l goes up by $\left(\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^{-\alpha}(e^{z}A)^{-1+\alpha}$ units. The higher is the technology, (entering through A and e^{z}), the less l has to go up.

Using (30) and (9), we get

$$\begin{aligned}
k &= l\frac{w}{r}\frac{\alpha}{1-\alpha} \\
&= y\left(\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^{-\alpha}(e^{z}A)^{-1+\alpha}\frac{w}{r}\frac{\alpha}{1-\alpha} \\
k(y) &= y\left(\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^{1-\alpha}(e^{z}A)^{\alpha-1} && (10) \\
k'(y) &= \left(\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^{1-\alpha}(e^{z}A)^{\alpha-1}
\end{aligned}$$

So the minimized total costs of producing output y with r, w, A, z is

$$\psi(y) = wl(y) + rk(y),$$

so the marginal costs are

$$\begin{aligned}
\psi' &= wl'(y) + rk'(y) \\
&= w\left(\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^{-\alpha}(e^{z}A)^{-1+\alpha} + r\left(\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^{1-\alpha}(e^{z}A)^{\alpha-1} \\
&= (e^{z}A)^{\alpha-1}\left[w\left(\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^{-\alpha} + r\left(\frac{w}{r}\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\right] \\
&= (e^{z}A)^{\alpha-1}w^{1-\alpha}r^{\alpha}\left[\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} + \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\right] \\
MC(y) &= (e^{z}A)^{\alpha-1}w^{1-\alpha}r^{\alpha}\left[\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\right] && (8)
\end{aligned}$$

B.1.2 Aggregate implications of individual production firm behaviour derivations

Now I derive the aggregate output as a function of aggregate factor inputs. Note that all production firms have in general different productivities and the aggregate production function is then a weighted average of individual production functions. Here I derive the weights.

The supply side condition on optimal capital/labour ratio in production is $\forall i$

$$K_{it} = l_{it}\frac{W_t}{r_t}\frac{\alpha}{1-\alpha}$$

which by summing over all industries, dividing by N and normalising by $\tilde{A}_t$ gives

$$k_t = l_t\frac{w_t}{r_t}\frac{\alpha}{1-\alpha} \tag{57}$$

With the aggregate price level normalisation, equation (16) becomes $1 = W_t^{1-\alpha}r_t^{\alpha}\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\tilde{A}_t^{-(1-\alpha)}$ which leads to

$$1 = w_t^{1-\alpha}r_t^{\alpha}\alpha^{-\alpha}(1-\alpha)^{\alpha-1}$$

which can be transformed into

$$\begin{aligned}
1 =& w_t\left(\frac{w_t}{r_t}\right)^{-\alpha}\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\\
\left(\frac{w_t}{r_t}\right)^{\alpha} =& w_t\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\\
\frac{w_t}{r_t} =& w_t^{1/\alpha}\alpha^{-1}(1-\alpha)^{\frac{\alpha-1}{\alpha}}
\end{aligned} \tag{58}$$

this combined with the previous results gives

$$\begin{aligned}
k_t &= l_t w_t^{\alpha}\frac{1}{\alpha}(1-\alpha)^{\frac{\alpha-1}{\alpha}}\frac{\alpha}{1-\alpha}\\
&= l_t w_t^{\frac{1}{\alpha}}(1-\alpha)^{\frac{-1}{\alpha}}\\
w_t &= (1-\alpha)\left(\frac{k_t}{l_t}\right)^{\alpha} \qquad (59)\\
r_t &= w_t^{\frac{\alpha-1}{\alpha}}\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}\\
r_t &= (1-\alpha)^{\frac{\alpha-1}{\alpha}}\left(\frac{k_t}{l_t}\right)^{\alpha-1}\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}\\
r_t &= \alpha\left(\frac{l_t}{k_t}\right)^{1-\alpha} \qquad (60)
\end{aligned}$$

which is the same as in other RBC models with perfect competition.

Relative price

$$\frac{P_{it}}{P_t} = \frac{\frac{1}{(\tilde{A}_{it})^{1-\alpha}} W^{1-\alpha} r^{\alpha} \alpha^{-\alpha} (1-\alpha)^{\alpha-1}}{W^{1-\alpha} r^{\alpha} \alpha^{-\alpha} (1-\alpha)^{\alpha-1} \left[\frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{\tilde{A}_j^{1-\alpha}} \right)^{1-\theta} \right]^{\frac{1}{1-\theta}}}$$

$$= \frac{\frac{1}{(\tilde{A}_{it})^{1-\alpha}}}{\left[\frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{\tilde{A}_{jt}^{1-\alpha}} \right)^{1-\theta} \right]^{\frac{1}{1-\theta}}} = \tilde{A}_{it}^{\alpha-1} \left[\frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{\tilde{A}_j^{1-\alpha}} \right)^{1-\theta} \right]^{\frac{1}{\theta-1}}$$

$$\frac{P_i}{P_t} = \tilde{A}_{it}^{\alpha-1} \left[\frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{\tilde{A}_i^{1-\alpha}} \right)^{1-\theta} \right]^{\frac{1}{\theta-1}} = \tilde{a}_{it}^{\alpha-1} \tag{61}$$

From the CES demand, it follows that the demand for output follows $Y_{it} = Y_t \left(\frac{P_{it}}{P_t} \right)^{-\theta}$ so the normalisation and the application of relative price result gives

$$y_{it} = y_t \tilde{a}_{it}^{\theta(1-\alpha)}$$

The normalised production function of individual production firm is

$$\begin{aligned}
Y_{it} =& K_{it}^{\alpha} (A_{it} l_{it})^{1-\alpha} \\
\frac{Y_{it}}{\tilde{A}_t} =& \left(\frac{K_{it}}{\tilde{A}_t} \right)^{\alpha} \left(\frac{A_{it}}{\tilde{A}_t} l_{it} \right)^{1-\alpha} \\
y_{it} =& k_{it}^{\alpha} \left(\frac{(1+e_{it})\tilde{A}_{it}}{\tilde{A}_t} l_{it} \right)^{1-\alpha} \\
y_{it} =& (1+e_{it})^{1-\alpha} k_{it}^{\alpha} \left(\tilde{a}_{it} l_{it} \right)^{1-\alpha}
\end{aligned}$$

which can be solved for capital

$$k_{it} = \frac{y_{it}^{\frac{1}{\alpha}}}{[(1+e_{it})\tilde{a}_{it} l_{it}]^{\frac{1-\alpha}{\alpha}}}$$

and combining it with the solution for capital from the optimal capital/labour $k_{it} = l_{it} \frac{w_t}{r_t} \frac{\alpha}{1-\alpha}$ ratio gives

$$\begin{aligned}
l_i \frac{w}{r} \frac{\alpha}{1-\alpha} =& \frac{y_{it}^{\frac{1}{\alpha}}}{[(1+e_{it})\tilde{a}_{it} l_{it}]^{\frac{1-\alpha}{\alpha}}} \\
y_{it}^{\frac{1}{\alpha}} =& l_{it}^{1+\frac{1-\alpha}{\alpha}} \frac{w_t}{r_t} \frac{\alpha}{1-\alpha} [(1+e_{it})\tilde{a}_{it}]^{\frac{1-\alpha}{\alpha}} \\
y_{it} =& l_{it} \left[\frac{w_t}{r_t} \frac{\alpha}{1-\alpha} \right]^{\alpha} [(1+e_{it})\tilde{a}_{it}]^{1-\alpha}
\end{aligned}$$

substitution the aggregate output produces

$$
\begin{aligned}
y_t \tilde{a}_{it}^{\theta(1-\alpha)} =& l_i \left[\frac{w_t}{r_t}\frac{\alpha}{1-\alpha}\right]^{\alpha} \left[(1+e_{it})\tilde{a}_{it}\right]^{1-\alpha} \\
y_t \left[\frac{w_t}{r_t}\frac{\alpha}{1-\alpha}\right]^{-\alpha} =& l_{it}(1+e_{it})^{1-\alpha}\tilde{a}_{it}^{(1-\alpha)(1-\theta)} \\
l_{it} =& y_t \left[\frac{w_t}{r_t}\frac{\alpha}{1-\alpha}\right]^{-\alpha} (1+e_{it})^{-1+\alpha}\tilde{a}_{it}^{-(1-\alpha)(1-\theta)} \\
\frac{1}{N}\sum_{i=1}^{N} l_{it} =& \frac{y_t}{\left[\frac{w}{r}\frac{\alpha}{1-\alpha}\right]^{\alpha}}\frac{1}{N}\sum_{i=1}^{N}(1+e_{it})^{-1+\alpha}\tilde{a}_{it}^{(-1+\alpha)(1-\theta)} \\
y_t =& \frac{l_t\left[\frac{w}{r}\frac{\alpha}{1-\alpha}\right]^{\alpha}}{\frac{1}{N}\sum_{i=1}^{N}\left[(1+e_{it})\tilde{a}_{it}^{1-\theta}\right]^{-1+\alpha}}
\end{aligned}
$$

and using the result $k = l\frac{w}{r}\frac{\alpha}{1-\alpha}$ again finally leads to

$$
y_t = \frac{k_t^{\alpha} l_t^{1-\alpha}}{\frac{1}{N}\sum_{i=1}^{N}\left[(1+e_{it})\tilde{a}_{it}^{1-\theta}\right]^{-1+\alpha}}
$$

in the symmetric case ($\tilde{a}_i = 1$ and $e_i = \bar{e}$, $\forall i = 1\dots, N$), I get

$$
y_t = k_t^{\alpha}\left[(1+\bar{e})l_t\right]^{1-\alpha}
$$

and $A_{it} = (1+e_{it})\tilde{A}_{it}$ so this would correspond to standard labour augmenting production function. Higher innovation step $\bar{e}$ hence increases the output.

B.2 Comparison to monopolistic pricing

The monopolistic price would be computed as the solution to the following problem: $\max_y p(y)y - \psi(y)$. The first order condition is $p'(y)y + p = \psi'$, which can be rewritten as

$$
\psi' = p\left(p'\frac{y}{p}+1\right) = p\left(\frac{\frac{\partial p}{\partial y}}{\frac{p}{y}}+1\right) = p\left(\frac{1}{\theta}+1\right) = p\frac{\theta+1}{\theta} = p\frac{1}{\mu}
$$

because $\frac{\mathrm{d}p}{\mathrm{d}y}/\frac{p}{y}$ is the price elasticity of demand. This leads to a standard mark-up result; the price a monopolist charges is a fixed mark-up over the marginal costs $p = \mu\psi'$.

If the new firm was to charge the monopolist price which would be above the limit price, then the incumbent firm could enter the market and realise profits. However, if the monopolist price is *lower* than the limit pricing price, then the new firm would maximize its profits charging the monopoly price, as at this price the incumbent could not enter the market anyway. In algebraic terms, the assumption requires that $\mu = \frac{\theta+1}{\theta} > (1+e_i)^{1-\alpha}$, or that the technological growth is *slow enough* so the new entrant finds it profitable to charge a limit price and drive out the incumbent rather than to apply the monopolistic price.

This implies that while the production firms are monopolists, due to the presence of a potential competitor who, despite being less productive, can potentially enter the market, the price mark-ups are smaller than in standard Calvo-type new keynesian models. This means that the prices are lower and the output is higher.

C Equilibrium conditions

Here I solve for the Euler equation in terms of the state variables and the policy functions l and $\tilde{l}$. The sequence is following:

1. solve for w,

$$w_t = (1-\alpha)\left(\frac{k_t}{l_t}\right)^{\alpha}$$

2. solve for c, normalised household first order labour supply condition

$$\begin{aligned} c_t &= \frac{\gamma-1}{\phi}(1-(l_t+\tilde{l}_t))w_t \\ &= \frac{(1-\alpha)(\gamma-1)}{\phi}(1-(l_t+\tilde{l}_t))\left(\frac{k_t}{l_t}\right)^{\alpha} \end{aligned}$$

   this can be re-written as

$$c_t = k_t^{\alpha}\frac{(1-\alpha)(\gamma-1)}{\phi}\left(\frac{1-(l_t+\tilde{l}_t)}{l_t^{\alpha}}\right)$$

   and this implies that

$$\frac{\partial c_t}{\partial l_t} = k_t^{\alpha}\frac{(1-\alpha)(\gamma-1)}{\phi}\left(-\frac{1+\alpha(1-l_t-\tilde{l}_t)l_t^{-1}}{l_t^{\alpha}}\right) < 0$$

   This means that people work more when they consume less, i.e. leissure and consumption are substitutes.

3. solve for y,

$$y_t = \frac{k_t^{\alpha} l_t^{1-\alpha}}{\frac{1}{N}\sum_{i=1}^{N}\left[(1+e_{it})\tilde{a}_{it}^{1-\theta}\right]^{-1+\alpha}}$$

4. solve for k_{t+1}, using the fact that in equilibrium all the output goes to the household:

$$Y_t = \frac{1}{N}\left[r_t\sum_{i=1}^{N}K_{it} + W_t\sum_{i=1}^{N}l_{it} + \sum_{i=1}^{N}s_{it}^{P}\Pi_{it}\right]$$

and the fact that the research firm spends all raised equity on labour,

$$W_t\tilde{l}_{it} = Q^R_{it}, \qquad \forall i = 1, \ldots, N$$

and the fact that the equilibrium holding of the stocks of the production firm is fixed $s^P_{it} = s^P_{it+1}$ (so there is no income from selling or expenditures from buying more production stocks), leads to

$$k_{t+1} = \frac{1}{g_{t+1}}(y_t + (1-\delta)k_t - c_t)$$

where

$$\begin{aligned} g_{t+1} = \frac{\tilde{A}_{t+1}}{\tilde{A}_t} &= \frac{\left[\frac{1}{N}\sum_{i=1}^N \left[(1+e_{it+1})\tilde{a}_{it}\right]^{(\alpha-1)(1-\theta)}\right]^{\frac{1}{(\alpha-1)(1-\theta)}}}{\left[\frac{1}{N}\sum_{i=1}^N \tilde{a}_{it}^{(\alpha-1)(1-\theta)}\right]^{\frac{1}{(\alpha-1)(1-\theta)}}} \\ &= \left[\frac{\sum_{i=1}^N \left[(1+e_{it+1})\tilde{a}_{it}\right]^{(\alpha-1)(1-\theta)}}{\sum_{i=1}^N \tilde{a}_{it}^{(\alpha-1)(1-\theta)}}\right]^{\frac{1}{(\alpha-1)(1-\theta)}} \end{aligned}$$

hence

$$\begin{aligned} k_{t+1} = \left[\frac{\sum_{i=1}^N \left[(1+e_{it+1})\tilde{a}_{it}\right]^{(\alpha-1)(1-\theta)}}{\sum_{i=1}^N \tilde{a}_{it}^{(\alpha-1)(1-\theta)}}\right]^{\frac{1}{(\alpha-1)(1-\theta)}} &\left\{\frac{k_t^\alpha l_t^{1-\alpha}}{\frac{1}{N}\sum_{i=1}^N \left[(1+e_{it})\tilde{a}_{it}^{1-\theta}\right]^{-1+\alpha}} + (1-\delta)k_t\right. \\ &\left. -\frac{(1-\alpha)(\gamma-1)}{\phi}(1-(l_t+\tilde{l}_t))\left(\frac{k_t}{l_t}\right)^\alpha\right\} \end{aligned}$$

D Impulse response functions

Note that a *good* realisation of μ_{it} leads to subsequent fall in many variables. Lets explain the logic behind this result on *normalised* variables using capital; good realisation of μ leads to innovation which increases the the level of productivity $\tilde{A}$. Even if the not normalised level of capital K stays the same, the normalised $k = K/\tilde{A}$ falls because of the increase in $\tilde{A}$. Only subsequently the capital start to grow back to the steady state level.

Variables l (total labour) and lp (labour in production sectors), lr labour in research sectors, g (growth rate of aggregate productivity index $\tilde{A}$) and e1 (innovation jump in first industry) are not normalised because they do not grow along the balanced growth path. Variables g and $e1$ show that in the current parametrisation there is not enough internal amplification.

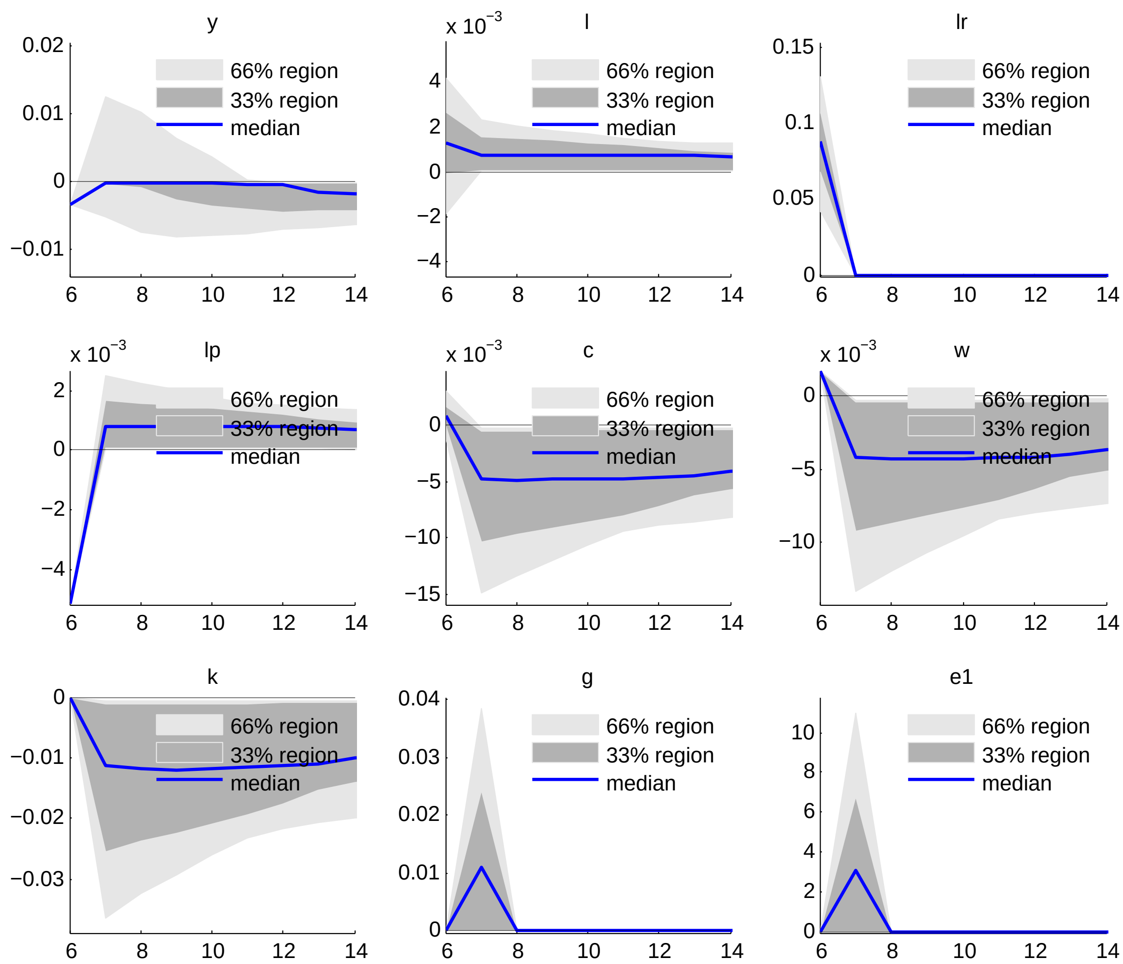

Figure 7: IRF on normalised variables, shock to quality of project of the first industry in period 6 ($\mu_{1,6}$)

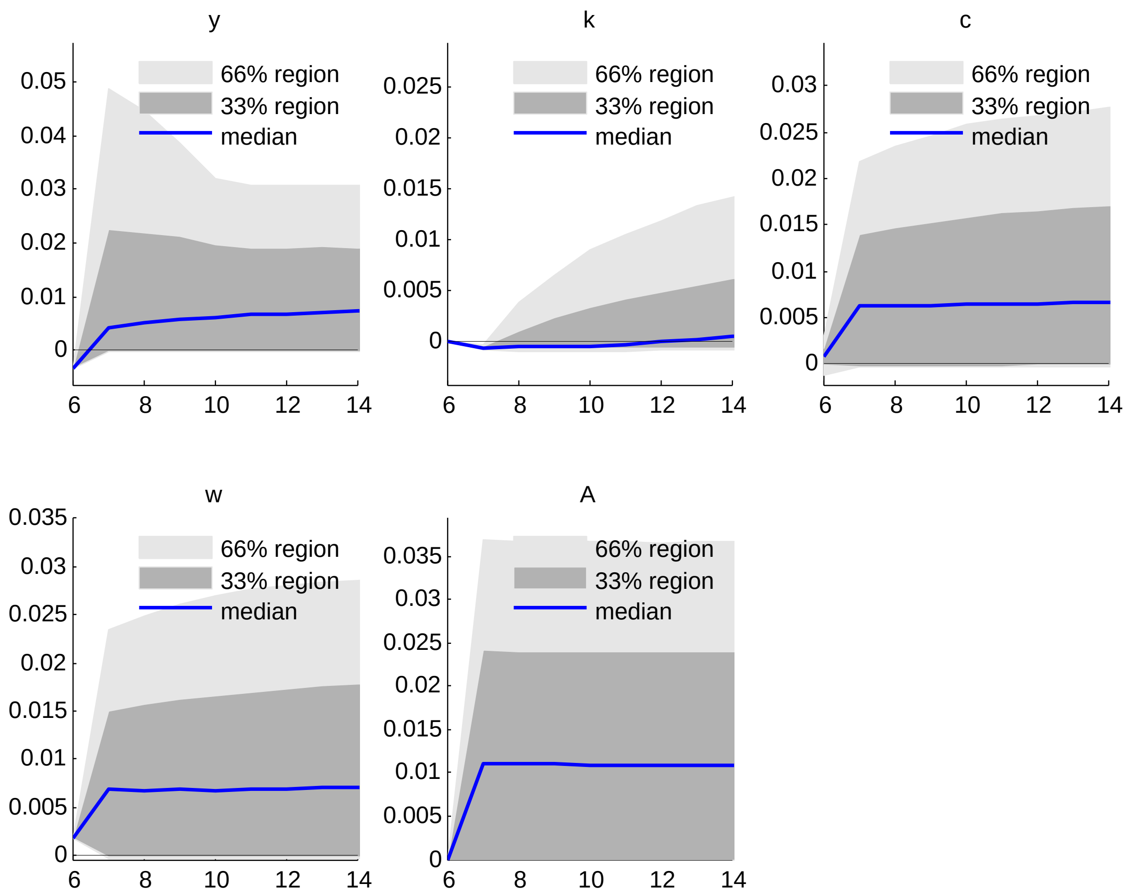

Figure 8: IRF, de-normalized

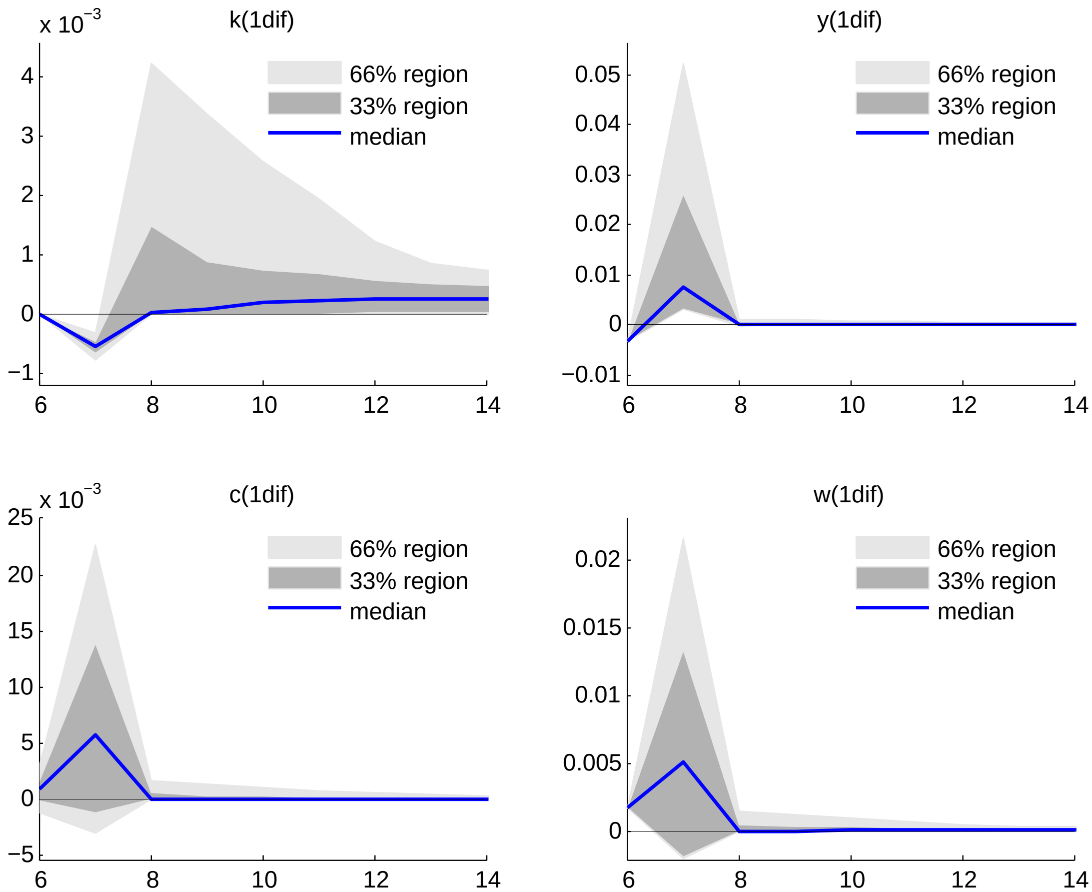

Figure 9: IRF, de-normalized and first differenced

y_{t-1}	y_t	c_{t-1}	c_t	c_{t+1}	l_{t-1}	l_t	l_{t+1}	i_{t-1}	i_t	i_{t+1}	qP_{t-1}	qP_t	qP_{t+1}	qR_{t-1}	qR_t	qR_{t+1}
1	0.6	0.8	0.7	0.5	0.6	0.3	0.1	0.2	0.1	0.0	0.7	0.3	0.1	-0.2	-0.0	-0.0
	1	0.6	0.8	0.7	0.3	0.6	0.3	0.0	0.2	0.1	0.3	0.7	0.3	0.1	-0.2	-0.0
		1	0.7	0.5	0.3	0.3	0.2	0.0	0.1	0.0	0.2	0.2	-0.0	0.2	-0.1	-0.0
			1	0.7	0.3	0.3	0.3	0.0	0.0	0.1	0.3	0.2	0.2	0.0	0.2	-0.1
				1	0.2	0.3	0.3	0.0	0.0	0.0	0.2	0.3	0.2	0.0	0.0	0.2
					1	0.4	0.2	0.2	0.0	0.0	0.4	0.0	0.0	-0.6	-0.0	-0.0
						1	0.4	0.0	0.2	0.0	-0.1	0.4	0.0	0.2	-0.6	-0.0
							1	-0.0	0.0	0.2	-0.1	-0.1	0.4	0.1	0.2	-0.6
								1	-0.0	-0.1	0.2	0.0	-0.0	-0.2	-0.0	-0.0
									1	-0.0	0.0	0.2	0.0	0.0	-0.2	-0.0
										1	0.0	0.0	0.2	0.0	0.0	-0.2
											1	0.3	0.1	-0.6	0.0	0.0
												1	0.3	0.1	-0.6	0.0
													1	0.0	0.1	-0.6
														1	-0.1	-0.1
															1	-0.1
																1.0

Table 4: Correlations in the simulated data (HP filtered)