Supply and Demand Function Equilibrium: a model of market power for input-output economies*

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Abstract

I propose a model of competition in input output networks which allows to dispense from assumptions that impose a lot of structure on which agents have more market power, such as which firm sets which prices (as in Bertrand/Cournot settings), or what the relative market power is (as in a Nash barganing setting). The model translates the language of double multi-unit auctions, adopted in Finance and IO, into the setting of an input-output economy. I show that under a tractable quadratic technology an equilibrium exists, and is computationally feasible. Moreover, I show that the price impact has a clear interpretation in terms of network quantities.

1 Introduction

Production of goods in modern economies typically features long and interconnected supply chains. Economic theory and basic intuition suggest that these connections are important in understanding relative market power of

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firms: the amount of surplus a firm is able to gather from the transactions. In turn, as a growing literature is showing, the interconnections are important in understanding of micro level distortions translate into aggregate outcomes. This paper's goal is to provide a model fit to study how the production network affects firms' market power.

In sections 2 and 5 I will claim through two examples that common price/quantity setting models impose a lot of structure on the distribution of market power, because the modeler has a lot of freedom in deciding which agent is setting which price. For example, in a tree network, if the model is built with firms setting the output quantity, upstream firms will tend to have larger markup, while if firms set the input quantity, downstream firms will tend to have a larger mark*down*.

To obtain a robust prediction, keeping the non-cooperative nature of competition, I set up the model as a double auction in which firms' choice variables are a profile of Supply and Demand schedules, in the spirit of the finance literature started by Kyle (1989). This allows to endogenize not only prices and quantities but also demand and supply slopes, that is, in financial terminology, the *price impact* of agents. This means that the modeler is much more agnostic on the structure it is imposing on the economy.

Unilateral competition in supply functions is a well-studied area of Industrial Organization, starting with Klemperer and Meyer (1989), but bilateral models are rare, with the exceptions discussed in the literature. Bilateral competition, via the double auction model, is instead very common in the finance literature, and due to the restrictive assumptions needed for close form solutions in both worlds (leading essentially to linear schedules), bears many formal analogies with the Supply Function Equilibria from I.O. From a technical point of view, the closest paper to mine is Malamud and Rostek (2017), which studies trade in interconnected markets and has formal analogies with my results. In their model though different trades involve different assets, and as such is missing one crucial feature of supply chain: the fact that inputs have to be bought to sell output, and so trading decisions are intertwined. Formally, my paper can be seen as a generalization of their model to an input-output setting.

To the best of my knowledge I am the first to apply these techniques to the analysis of full input-output economies.

Related literature For the purpose of this review it is convenient to divide the literature on market power in networks in three parts: the macro/sector level approach, the sequential competition, and the cooperative approach.

In the macro literature papers typically assume that either the markup is an exogenously given *wedge* between prices and marginal costs (Baqaee and Farhi (2017), Huremovic and Vega-Redondo (2016)), or is determined endogenously through monopolistic or oligopolistic competition. This means that firms internalize to some extent the effect of their price or quantity decisions on their customers. An issue here arises, because of course the behavior of customers will depend in turn on their reaction and strategic decisions, and so on down all the network. To make the model tractable Grassi (2017), De Bruyne et al. (2019), Baqaee (2018), Kikkawa et al. (2019) solve firms' price (or quantity) setting problem under the assumption that the output of customers be taken as given, so that the interactions other than one step links can be abstracted away. This has the effect of *constraining* markups and market power to depend on sector level characteristics, such as the elasticity of substitution, the number of firms, the distribution of productivity (Grassi (2017)), or the specific input share in customer technology (Kikkawa et al. (2019)). These models abstract from the strategic behavior of firms that take into account their position in the supply chain and behave accordingly. The assumption that firms internalize the reaction of first neighbor and not anyone else might not be ideal if firms are large with respect to their own market, or belong to a supply chain which is crucial for their business. In both these cases, we would expect that firms hold a lot of information to what happen downstream from them, and that they would profit by doing so.

An exception is Acemoglu and Tahbaz-Salehi (2020), that builds a model in which firms solve the market clearing conditions in a way similar to mine, but where prices are determined at the link level through a Nash bargaining solution. This means that relative market power, though affected by the network, will be crucially affected by the choice of bargaining parameters.

These mechanisms are present in the micro literature, when a supply chain is present, in the almost universal form of sequential competition. *Competition* is meant as a generic term for price setting (as in Hinnosaar (2019), Federgruen and Hu (2016)), quantity setting, or more complicated procedures such as auctions (Kotowski and Leister (2019)). This methodology has the discussed drawback of imposing a lot of structure on which firms have largest market power. Moreover it is feasible only for acyclic networks. The relevance of cycles in real production networks is not yet very clear, but on strict terms they are certainly not acyclic[1].

A close paper that uses a sequential competition model is Carvalho et al.

[1]For example, Tintelnot et al. (2018) estimates that no more than 23% of the links in the Belgian firm to firm production network are in violation of acyclicity. This might justify neglecting acyclicity as a first approximation, but is a number distant from 0.

(2020), that builds a tractable model to identify "bottlenecks" in real production network data. In their terminology bottlenecks are those firms that, if removed, would increase welfare. In their world, links have exogenous capacity constraints. To the extent that we expect these capacity constraints to represent firms underproduction due to market power, my model can be seen as a way to endogenize the capacity of each link[2]: each firm chooses that optimally as a solution to its oligopoly problem, and the capacity in equilibrium will reflect the market power.

Some papers study the interconnection of final markets of different firms, and as such do not need to address the input-output issues. In this category falls Bimpikis et al. (2019).

The use of supply schedules as choice variables was introduced in Klemperer and Meyer (1989), and was popularized in applications especially in the context of electricity markets in Green and Newbery (1992). A recent review is in Delbono and Lambertini (2018). These studies feature market power on one side of the market only, as typical in oligopoly models. The papers that have dealt with the problem of bilateral oligopoly, allowing for market power on both sides of the market, are Weretka (2011), that attacks the problem constraining the schedules to be linear (instead of getting this as an equilibrium result), thus gaining traction in the analysis for general functional forms, while Hendricks and McAfee (2010) limit the players to a choice of a parameter of given supply and demand schedules, leaving again to the modeler *de facto* to have to choose relative market power. The closest paper to mine, as discussed above, is Malamud and Rostek (2017), that analyses a game in supply and demand schedules for decentralized trading. Some of the intuition and mathematical characterizations are similar to this paper, even if the setting is different.

Another side of the literature is composed by models that employ cooperative tools, such as stability and matching. The literature started by Hatfield et al. (2013) uses as primitives *bilateral trades*, that is pairs of quantities and prices, so that this is jointly a model of trading and of network formation. Differently from my paper, they consider indivisible goods (trades are discrete). Also in this setting dealing with cycles is complicated: Fleiner et al. (2019) introduces a stability concept weaker than stable matching to deal with general networks. Fleiner et al. (2019) studies the model in presence of frictions.

My paper is also connected to an older line of literature, generally called "general oligopolistic competition", studying how to represent a full econ-

[2] My model cannot be seen as an exact generalization because the production function I employ is different.

omy with interconnected trades as a game. This literature (for a review see Bonanno (1990)) has underlined the conceptual complexity of defining a concept of equilibrium in an economy with market power and fully strategic agents. One of the routes taken has been to reduce the problem to game theory by defining a Cournot-Walras (or analogously, Bertrand-Walras) equilibrium. This has proven to be a hard problem: Dierker and Grodal (1986) showed that an equilibrium might not exist, not even in mixed strategies, due to lack of quasiconcavity of the payoff functions. As examples of this literature, Gabszewicz and Vial (1972) or Marschak and Selten (2012) work with general technologies and any number of consumers, but assume either that there are no intermediate inputs or that the price maker firms do not sell among themselves. The closer in spirit is Nikaido (2015), who uses the market clearing conditions to back up quantities as functions of prices, but his method is limited to Leontief technology, and Benassy (1988) which defines an objective demand by means of a fixprice equilibrium, thus not limiting himself to constant returns technology, but as a drawback having to contemplate a rationing rule, and losing a lot in terms of tractability. These methods are the analogous in their setting of the residual demand in 4.2. My model, though restrictive in the technology assumption it can deal with, is a game, thus fully strategic.

The rest of the paper is organized as follows: section 2 illustrates the model applied to a simple supply chain, where results are particularly sharp. Section 3 presents the model in full generality. Section 5 discusses how the network affects market power through some examples. Section 6 explains how it is possible to solve the model numerically and how feasible it is. Section 7 concludes.

2 Example: a simple supply chain

Consider a simple supply chain: there are three sectors, 0, 1 and 2, respectively composed by n_0, n_1 and n_2 firms. Firms in sector 0 are final good producers and sell directly to the consumer. Firms in sectors 1 and 2 are intermediate goods producers that employ only labor. The situation is illustrated in Figure **??**:

Which is the sector where a merger causes a larger efficiency loss? the simplest modeling strategy in this setting is a *sequential oligopoly*[3]. To fix ideas assume that goods in each sector are perfect substitutes, and at each stage of the supply chain firms compete à la Cournot, taking as given the

[3]Used in various combinations of price and quantity competition e.g. by Ordover et al. (1990) and Salinger (1988)

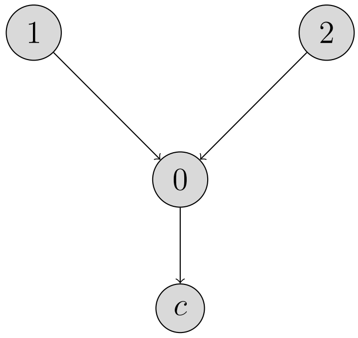


Figure 1: A simple supply chain. Sector 1 and 2 firms sell their output to sector 0 firms, who in turn sell to consumers, denoted by c.

input price they face. In our setting this means that firms in sectors 1 and 2 play first, simultaneously, committing to supply a certain quantity. Then firms in sector 0 do the same, taking the price of good 1 and 2 as given. The model can then be solved by backward induction[4].

However, if there is no compelling physical reason for assuming firms in sector 1 and 2 have precedence over sector 0, an equally reasonable option would be to assume that firms commit to *input* quantities (prices) rather than their output equivalents. An analogous model can then easily be constructed assuming firms in sector 0 decide first, then firms in sector 1 and 2. The purpose of this section is to show that this freedom of choice is problematic because can crucially affect predictions.

In order to make things precise, one must specify a labor (or primary input) supply function that sector 1 and 2 can be oligopolists over. In order to do this, and provide a unified setting, assume that there is a continuum of representative consumers with utility function $U(Q_2, L2) = \sqrt{Q_0} - L$, where Q_0 is the good produced by sector 0, and L is labor (or any primary input owned by agents). Since they are a continuum, they take prices as given, that yield the following expression for the (inverse) demand for good 0:

$$\frac{p_0}{w} = Q_0^{-\frac{1}{2}} \tag{1}$$

where π is the profit, which is rebated to the consumers.

Equivalently, through the budget constraint, we can write an (inverse) labor supply:

$$\frac{w}{p_0} = L + \frac{\pi}{p_0} \tag{2}$$

[4]This is a "networked" version of the simplest setting e.g. in Salinger (1988)

Expressions 1 and 2 are equivalent (via the budget constraint).

Now we have two versions of the model, depending on whether *output* or *input* quantities are set:

Output-setting At $t = 0$ firms in sectors 1 and 2 decide their output quantity; at $t = 1$ firms in sector 0 do the same. Firms in sector 0 face the inverse demand function $\frac{p_0}{w} = Q_0^{-\frac{1}{2}}$ and all firms take their *input* prices as given. In particular w is taken as given by every agent and we can normalize it to 1;

Input-setting At $t = 0$ firms in sector 2 decide their input quantity; at $t = 1$ firms in sector 1 and 2 do the same. Firms in sectors 1 and 2 face the inverse labor supply function $\frac{w}{p_0} = L + \frac{\pi}{p_0}$ and all firms take their *output* prices as given. In particular p_0 is taken as given by every agent and we can normalize it to 1.

Now we can compare the welfare effect of mergers in the two versions through equilibrium consumer utility $U(Q_2, L2) = \sqrt{Q_0} - L$. Let us focus on the case in which the technology available to firms in sector 0 is Cobb-Douglas in goods 1 and 2, with production function: $f_0(q_1, q_2) = \sqrt{q_1 q_2}$. Sector 1 and 2 produce 1 unit of output with each unit of input (the production function is $f_i(q) = q$, $i = 1, 2.$). Firms in the same sectors are identical and produce perfectly substitutable goods[5]

The model can easily be solved numerically: Figure 2 plots the welfare losses from a merger in sector 1 and sector 0 in the two models, starting from a situation where all 3 sectors have the same number of firms $n_1 = n_2 = n_0$. We can immediately see that for $n \leq 2$ the choice of the model affects crucially which sector is more relevant to the policy maker: in the output-setting version sector 1 and 2 are the most important, while under the input-setting version sector 0 is the most important.

Why is this so? To fix ideas, focus on the output-setting game. In this sequential setup upstream firms perceive a smaller elasticity of demand, because for each variation in supplied quantity some of it will imply variation in final consumer price, but some of it will instead disappear in the markup adjustment of the downstream firms. And since upstream firms perceive larger elasticity they will tend to have larger markups. This effect is discussed formally and more in detail in Example 4. In the welfare computation other effects are present, for example an increase in markup of the downstream

[5] I focus on quantity setting because with price setting and perfect substitutes the outcome is the competitive one in either case (although Proposition 4 still holds trivially, with weak inequalities).

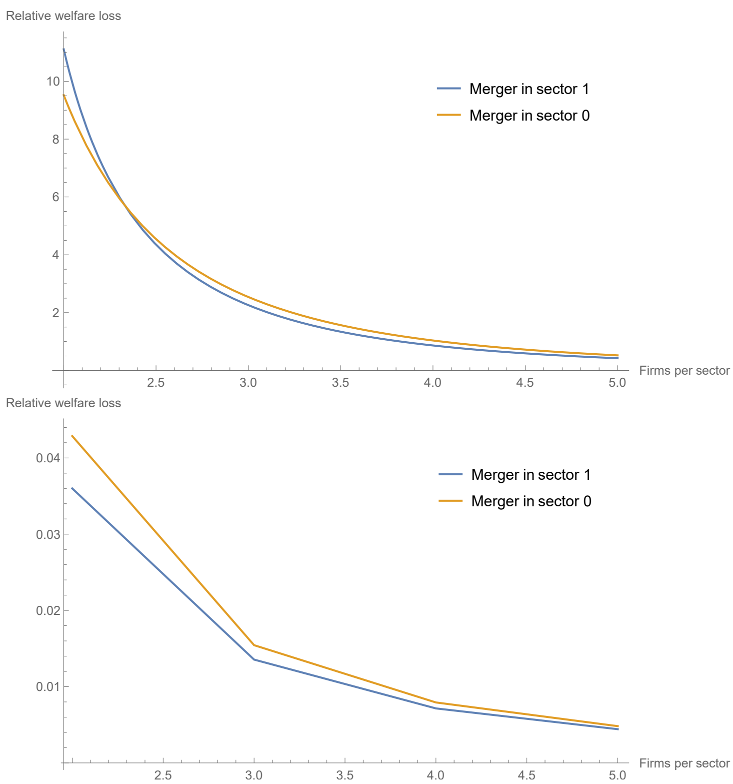


Figure 2: Comparison of welfare loss from a merger in sector 1 and 0, for the output-setting model (Above), and the input-setting one (Below).

sector will result in an increased market power for upstream firms that will react increasing their markup. The exact balance depends on the functional forms used and parameter values, and this is the reason why for $n > 2$ the property does not hold any more.

This example is useful to see exactly what goes wrong: the assumption of *one-sided* market power: if we focus on output-setting we have decreasing elasticity going upstream, if we focus on input-setting we have increasing elasticity going downstream! a model that solves this problem would allow firms to internalize to some extent *both* input supply and output demand elasticities.

How to solve this problem? if sector 1 were competitive, then we could consider firms in sector 0 that decide both input and output quantities, internalizing their effect not only on the (inverse) demand curve, but also on the supply curve. But if firms in sector 1 are a finite number and have market power this does not work. As anticipated in the Introduction, the solution I propose is to let all the firms commit not to a quantity or a price, but to a full *supply* and/or *demand* schedule, as detailed in the next sections. For the purpose of comparison with the models just described, I anticipate here the predictions of the model for the simple network analysed in this section. Figure 1 shows the welfare loss from mergers. We can see that sector 0 is predicted to be the most important, which is consistent with the intuition of it being non-substitutable, while sector 1 and 2 are[6].

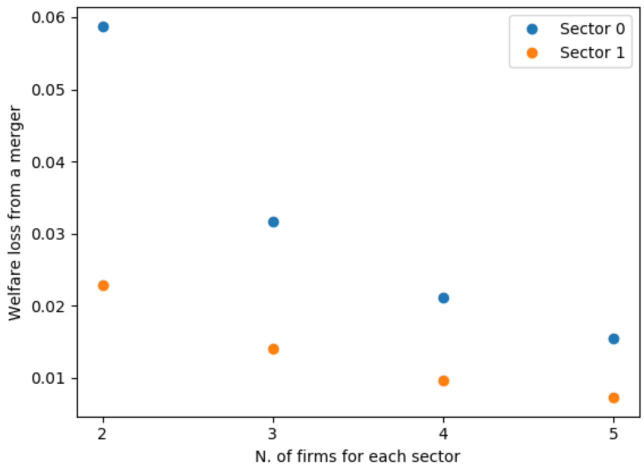
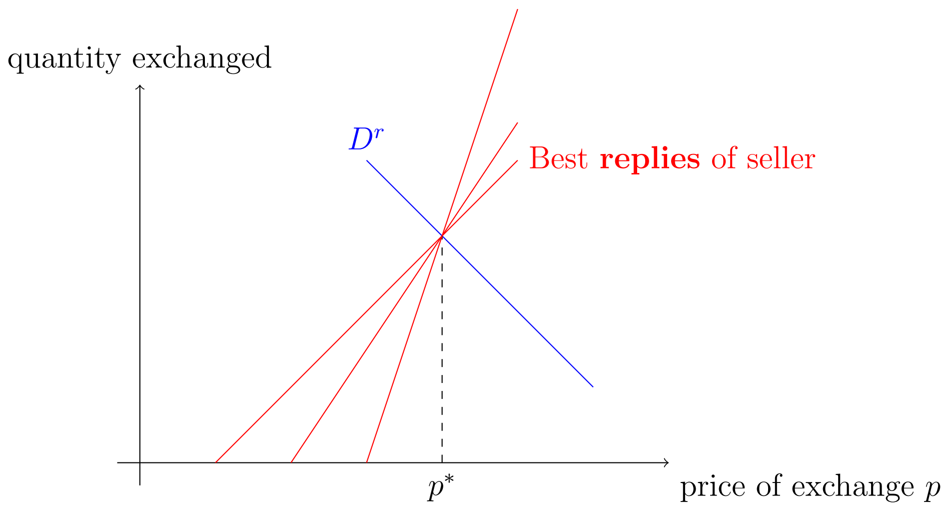


Figure 3: Welfare loss from a merger in sector 0 and 1 of the tree in Figure 1 as a function of the number of firms per sector.

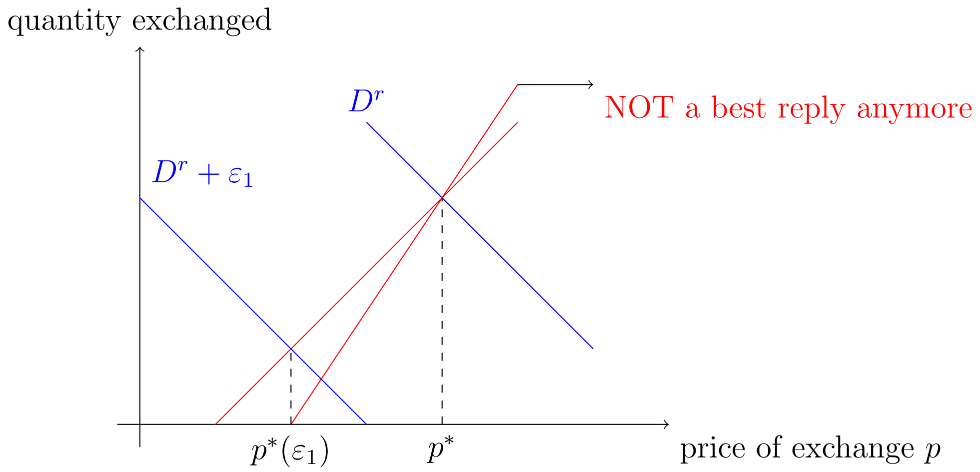
This type of modeling has a classical multiplicity problem, as illustrated by Figure 5. The solution, both in the oligopoly and in the market microstructure literature, consists in introducing some source of uncertainty, so that all feasible prices can be realized in equilibrium for some realizations of the uncertainty, and this pins down the full demand or supply schedules rather than just a point on them.

Differently from the Klemperer and Meyer (1989) setting, where a stochastic shock to the exogenous demand function is sufficient to pin down unique best replies, in a supply chain, or more generally in a network economy, more prices have to be determined. This means that the uncertainty in demand alone is not able any more to solve the multiplicity problem. In a network setting, we must add a source of uncertainty in every market, that is one

[6]Care must be taken here because the functional form for the technology that I use in the Supply and Demand Function Equilibrium model is not Cobb-Douglas, because does not yield a tractable solution. However, the same qualitative predictions as in Figures **??** can be obtained



(a) If optimal price for seller is p, all red lines represents best replies.



(b) Since the parameter ε_1 is stochastic, the seller will adjust its supply function in order to pin down the optimal price for *any* realization of ε_1, thereby destroying the multiplicity.

Figure 4: Multiplicity problem and solution in Supply Function Equilibrium.

for every price to be determined. To solve this issue, I am going to introduce a stochastic parameter ε in the technology of each sector. This will be analogous to the quite standard notion of a labor productivity shock, shifting the amount of good that a firm is willing to buy from its suppliers and simultaneously the quantity that it is willing to sell.

In almost all the cited papers, tractable solutions are obtained through specific functional form assumptions that lead to linear schedules. In our case it means we need a technology yielding a quadratic profit function. The key to achieving the parametric expression we need is introducing another input: labor. The expression of the production function for sector 1 is:

$$\Phi_1(q_{1\alpha}) = \min\{-\varepsilon_1 + \sqrt{\varepsilon_1^2 + l_{1\alpha}}, q_{1\alpha}\} \tag{3}$$

Note that:

1. the parameter ε_i can be seen as a shifter of the marginal product of labor;

2. at the optimum it must be the case that $-\varepsilon_i + \sqrt{\varepsilon_i^2 + 2l_{i\alpha}} = q_{i\alpha}\, l_{i\alpha}$, and solving this we get $l_{i\alpha} = \varepsilon_i q_{i\alpha} + \frac{1}{2}q_{i\alpha}^2$, which means that the labor productivity shock shifts linearly the amount of labor needed to transform a given quantity of input in output;

3. moreover, at the optimum the profit assumes a linear quadratic form:

   $$\pi_i^* = p_i \min\{-\varepsilon_i + \sqrt{\varepsilon_i^2 + l_{i\alpha}}, q_{i\alpha}\} - wl_{i\alpha} - p_{i-1}q_{i\alpha} = p_i q_{i\alpha} - w\frac{1}{2}q_{i\alpha}^2 - p_{i-1}q_{i\alpha}$$

   which is crucial to analytical tractability.

3 The Model

In this section I define the model in full generality, that is without making parametric assumptions on the technology and the consumer utility, to clarify the generality of the setting. In paragraph 3.2 I discuss the parametric assumptions needed for the subsequent analysis.

3.1 General setting

Firms and Production Network There are N economic sectors, each sector $i = 1, \ldots, N$ is populated by a finite number n_i of identical firms. Firms are denoted with greek letters $\alpha = 1, \ldots, n_i$. Each sector needs as inputs the goods produced by a subset $\mathcal{N}_i^{in}$ of other sectors, and sells its outputs to a subset of sectors $\mathcal{N}_i^{out}$. I denote the transformation function available to all the firms of sector i as Φ_i. This is a function of the input and output quantities, and also of a stochastic parameter ε_i that will have the role of a technological shock. I denote the joint distribution of $\varepsilon = (\varepsilon_i)_i$ as F.

I denote d_i^{out} the *out-degree* and d_i^{in} as the *in-degree* of sector i. Sectors are connected if one is a customer of the other. E is the set of existing connections, $E \subseteq N \times N$.

Inputs of firms in sector i are $q_{i\alpha,j}, j = 1, \dots d_i^{in}$ and outputs $q_{k,i\alpha}, k = 1, \dots d_i^{out}$. Each of the inputs of sector i has an *input-output weight* ω_{ij}, and the corresponding vector is denoted $\omega_i = (\omega_{i1}, \dots, \omega_{id_i^{in}})$. Firms in each sector may produce more than 1 good, but different sectors never produce the same good.

Sectors and connections define a weighted directed graph $\mathcal{G} = (N, E)$ which is the *input output network* of this economy. Its adjacency matrix is $\Omega = (\omega_{ij})_{i,j \in N}$. Note that in this model the input-output network is a *sector-level* concept.

Consumers Consumers are a continuum and identical, so that there is a representative consumer[7] The labor market is assumed competitive, that in particular means firms will have no power over the wage. Hence the wage plays no role, so we are going to assume that the labor is the numeraire good, and normalize it to 1 throughout. Similarly to the firms, I am going to assume that the consumer utility depends on stochastic parameters $\varepsilon_c = (\varepsilon_{i,c})_i$, one for each good consumed: $U(c, L, \varepsilon_{i,c})$. Denote the demand for good i derived by U as $D_{ci}(p_i, \varepsilon_c)$.

Notation I write $i \to j$ to indicate that a good produced by sector i is used by sector j in production (or equivalently that $(i, j) \in E$). I write p_i^{out} for the vector of all prices of ouputs of sector i, and p_i^{in} for the vector of input prices, and $p_i = \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix}$. u_i^{out} denotes a vector of ones of lenght d_i^{out}, while u_i^{in} a corresponding vector of ones of length d_i^{in}, and $\tilde{u}_i = \begin{pmatrix} u_i^{out} \\ -\omega_i \end{pmatrix}$. Similarly, I_i is the identity matrix of size $d_i^{out} + d_i^{in}$, while I_i^{in} and I_i^{out} have respectively size d_i^{in} and d_i^{out}.

[7] In particular, it is assumed that each infinitesimal consumer owns identical shares of all the firms so that we avoid the difficulties uncovered by Dierker and Grodhal: see the Introduction.

Unless specified differently, the inequality $B \geq C$ when B and C are matrices denotes the *positive semidefinite* (Löwner) ordering. That is: $B \geq C$ if and only if $B - C$ is positive semidefinite.

The Game The competition among firms take the shape of a game in which firms compete in supply and demand functions. This means that the players of the game are the firms, and the actions available to each firm α in sector i are vectors of supply and demand functions $(S_{k_1,i\alpha}, \ldots, S_{k_{d_i^{out}},i\alpha})$, $(D_{i\alpha,j_1}, \ldots, D_{i\alpha,j_{d_i^{in}}})$, $l_{i\alpha,j}(\cdot)$ defined over pairs of output price and realization of sector level stochastic parameter (p_i, ε_i).

The reason to introduce a stochastic parameter is that this type of modeling has a classical multiplicity problem, as illustrated by Figure 5. The solution, both in the oligopoly and in the market microstructure literature, consists in introducing some source of uncertainty, so that all feasible prices can be realized in equilibrium for some realizations of the uncertainty, and this pins down the full demand or supply schedules rather than just a point on them.

Differently from the Klemperer and Meyer (1989) setting, where a stochastic shock to the exogenous demand function is sufficient to pin down unique best replies, in a supply chain, or more generally in a network economy, more prices have to be determined. This means that the uncertainty in demand alone is not able any more to solve the multiplicity problem. In a network setting, we must add a source of uncertainty in every market, that is one *for every price* to be determined. That will be the role of the productivity shock, shifting the amount of good that a firm is willing to buy from its suppliers and simultaneously the quantity that it is willing to sell.

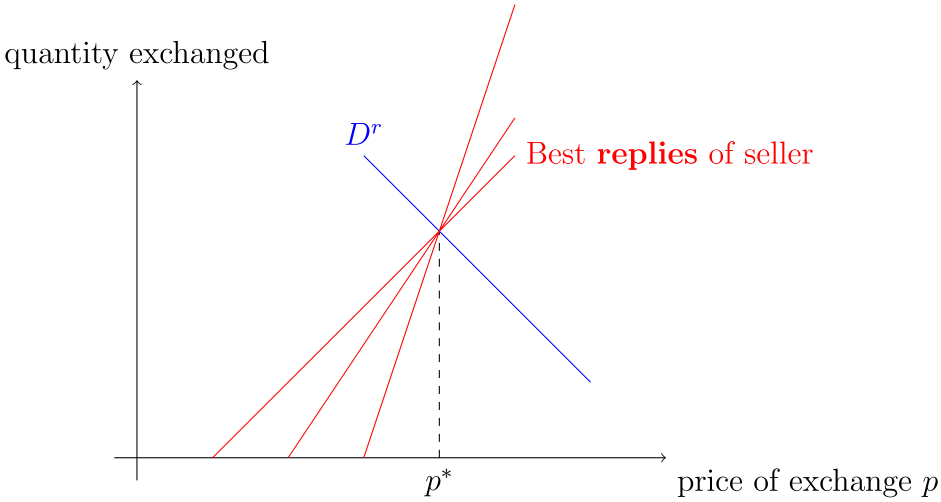
The feasible supply and demand schedules must satisfie:

i) they are nonnegative;

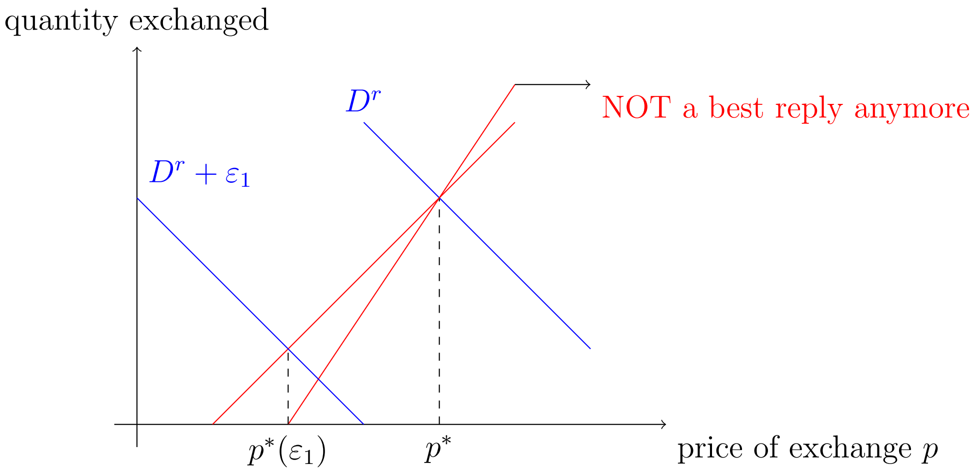
ii) they must satisfy the **technology constraint**, that is for any possible (p_i, ε_i):
$$\Phi_i(S_{i\alpha}(p_i, \varepsilon_i), D_{i\alpha}(p_i, \varepsilon_i), l_{i\alpha}(p_i, \varepsilon_i), \varepsilon_i) = 0 \tag{4}$$

iii) the maps $(S_{i\alpha}, D_{i\alpha})$ must be continuously differentiable and have Jacobian $J_{i,p_i^{out},-p_i^{in}}$ which is everywhere positive semidefinite and has rank at least $d_i - 1$ (the maximum minus 1)[8]; note that the differentiation is done with respect to the variables $(p_i^{out}, -p_i^{in})$;

[8]It might not be positive definite because the technology constraint implies, by the chain rule: $\nabla\Phi_{i,S} JS_i + \nabla\Phi_{i,D} JD_i + \nabla\Phi_{i,l} Jl_i = \mathbf{0}$. Depending on how labor enters the technology this might become a linear constraint on the rows of the Jacobian: it is indeed what happens under the parameterization introduced in 3.2, as will be clear in the following.



(a) If optimal price for seller is p, all red lines represents best replies.



(b) Since the parameter ε_1 is stochastic, the seller will adjust its supply function in order to pin down the optimal price for *any* realization of ε_1, thereby destroying the multiplicity.

Figure 5: Multiplicity problem and solution in Supply Function Equilibrium.

iv) they have a bounded support.

These conditions allow us to define the *realized prices* $p^*(\varepsilon)$ through the market clearing equations. The function p^* is the one implicitly defined by the market clearing equations:

$$\begin{aligned}\sum_\beta D_{j\beta,k}(p_j^{out}, p_j^{in}, \varepsilon_j) &= \sum_\alpha S_{k\alpha}(p_k^{out}, p_k^{in}, \varepsilon_k) \quad \text{if } k \to j \\ D_{ck}(p_{ck}, \varepsilon_{ck}) &= \sum_\beta S_{k\beta}(p_k^{out}, p_k^{in}, \varepsilon_k) \quad \text{if } k \to c \end{aligned} \tag{5}$$

To show that the regularity conditions indeed imply that the market clearing system can be solved, the crucial step is to show that they translate to regularity conditions on the Jacobian of the function whose zeros define the system above, and then a global form of the implicit function theorem (Gale and Nikaido (1965)) can be applied. This is done in the next Proposition.

Proposition 1. *The market clearing conditions define a function:*

$$p^* : \times_i \mathcal{E}_i \to \mathbb{R}_+^{|E|\times|E|}$$

Note that p^* is also differentiable, but since all the equilibrium analysis and hence the optimizations, will be performed in the linear case, we do not need this property: the goal of this section is simply to define the game.

Now that we built the prices implied by the players's actions, we can define the payoffs. These are the expected profits calculated in the realized prices p^*:

$$\pi_{i\alpha}(S_{i\alpha}, D_{i\alpha}, l_{i\alpha}) = \mathbb{E}\left(\sum_k p_{ki}^* S_{k,i\alpha} - \sum_j p_{ij}^* D_{i\alpha,j} - l_{i\alpha}\right) \tag{6}$$

where to avoid clutter I omitted to write each functional variable.

Hence, formally, the game played by firms is: $G = (I, (A_{i\alpha})_{(i,\alpha)\in I}, (\pi_{i\alpha})_{(i,\alpha)\in I}, F)$, where $I = \{(i,\alpha) \mid i = 1, 2,\ \alpha = 1, \dots n_i\}$ denotes the set of firms, and $A_{i\alpha}$ is the set of profiles of supply and demand functions that satisfie the assumptions above.

Example 1. The model by Klemperer and Meyer (1989) can be seen as a special case of this setting, in which there is only one sector and the network $\mathcal{G}$ is empty. Their setting is a "partial" equilibrium one, in which the consumers do not supply labor to firms but appear only through a demand function $D(\cdot)$, and firms have a cost function for production $C(\cdot)$, that does not explicitly represent payments to anyone. The strategic environment is precisely the same though: if in the one sector version of my model we assume that firms produce using labor through a production function $f(l) = C^{-1}(q)$ and the consumer utility gives rise to demand D, the game G played by firms is precisely the same as in Klemperer and Meyer (1989).

The welfare of the consumer is $U(C, L)$, where $C(p^*, \varepsilon) = (C_{ci,\alpha}(p_i, \varepsilon_i))_{i,\alpha}$ is the vector of quantities of goods consumed in equilibrium, and $L = \sum_{i,\alpha} l_{i,\alpha}(p_i^*, \varepsilon_i)$ is the total labor used in the economy[9]. The consumers, being atomic, take all prices as given and thus are a non-strategic component of the model, that enter in the game only through their aggregate demand function.

Supply and Demand Function Equilibrium To compute the predictions of the model I just need to specify the role of the stochastic parameters ε. I will use it as a selection device, as made formal by the next definition.

Definition 3.1. A Supply and Demand Function Equilibrium is a profile of prices and quantities of traded goods (p_{ij}, q_{ij}) for all $(ij) \in E$ that realize in a Nash Equilibrium of the game G for $F \xrightarrow{\mathcal{D}} 0$:

$$\begin{aligned} p_{ij} &= p_{ij}^*(0) \\ q_{ij} &= \sum_\alpha D_{i\alpha,j}^*(p_{ij}^*, 0) \quad \forall (i,j) \in E \end{aligned} \tag{7}$$

So in practice I am using the stochastic variation to “identify” the equilibrium schedules, but when computing the equilibrium predictions I am considering the case in which the shock vanishes.

3.2 Parametric Assumptions

To obtain a tractable solution, I adopt parametric assumptions on the technology. Since firms may produce more than 1 good, I have to express the technology via a *transformation function*. Specifically, assume that the production possibility set of each firm α in sector i be the set of $(q_{k,i\alpha})_k, (q_{i\alpha,j})_j, (l_{i\alpha,kj})_{k,j}$ such that there exists a subdivision $(z_{i\alpha,kj})$ of inputs satisfying $q_{i\alpha j} = \sum_k z_{i\alpha,kj}$, and:

$$q_{k,i\alpha} = \sum_j \omega_{ij} \min\{-\varepsilon_i + \sqrt{\varepsilon_i^2 + 2l_{i\alpha,kj}}, z_{i\alpha,kj}\} \quad k = 1, \ldots, d_i^{out} \tag{8}$$

[9] It is not necessary to impose a “labor market clearing” condition because it is redundant with the budget constraint of the consumer, consistently with the decision to normalize the wage to 1.

This functional form[10] turns out to be particularly convenient because at the optimum we must have $-\varepsilon_i + \sqrt{\varepsilon_i^2 + 2l_{i\alpha,kj}} = z_{i\alpha,kj}$, so that $l_{i\alpha,kj} = \varepsilon_i z_{i\alpha,kj} + \frac{1}{2}z^2_{i\alpha,kj}$, and the profit function becomes linear-quadratic:

$$\pi_i = \sum_k p_{ki} q_{k,i\alpha} - \sum_j p_{ij} q_{i\alpha,j} - \varepsilon_i \sum z_{i\alpha,kj} - \frac{1}{2} \sum z^2_{i\alpha,kj} \tag{9}$$

This will be crucial in achieving a linear equilibrium.

The analogous assumptions on the utility function of the consumer are that it be quadratic in consumption and (quasi-)linear in disutility of labor L:

$$U((c_i)_i, L) = \sum_i \frac{A_{i,c} + \varepsilon_{i,c}}{B_{c,i}} c_i - \frac{1}{2} \sum_i \frac{1}{B_{c,i}} c_i^2 - L$$

This means that the consumer has demands of the form: $D_{ci} = \max\left\{A_i - B_{c,i}\frac{p_{ci}}{w}, 0\right\}$.

4 Solution and Existence

In the following I will focus on S&D equilibria in symmetric linear schedules.

Definition 4.1. A Supply and Demand Function Equilibrium in *symmetric linear schedules* is a profile of functions $\boldsymbol{\sigma} = ((S_{i\alpha})_\alpha, (D_{i\alpha})_\alpha, (l_{i\alpha})_\alpha)_i$ defined on open sets $(\mathcal{O}^p_{i,\alpha})_{i,\alpha} \times (\mathcal{O}^\varepsilon_{i,\alpha})_{i,\alpha}$ such that:

i) $\boldsymbol{\sigma}$ is a Nash Equilibrium of the game G;

ii) (Symmetry) in each sector i firms play the same schedules: $D_{i\alpha} = D_i$, $S_{i\alpha} = S_i$, $l_{i\alpha} = l_i$, $\mathcal{O}^p_{i,\alpha} = \mathcal{O}^p_i$ and $\mathcal{O}^\varepsilon_{i,\alpha} = \mathcal{O}^\varepsilon_i$;

[10] A more classical choice, especially in the macro literature, is the one of a production function belonging to the Constant Elasticity of Substitution class. This does not yield tractable expressions here.

To allow for general input intensity parameters and more general substitution patterns is possible in the present framework, using a variation that leads to the profit function (in the single output case, for simplicity):

$$p_i \sum_j \omega_{ij} q_{ij} - \sum_j p_{ij} q_{ij} - \frac{1}{2} \sum q_{ij}^2 - \frac{\sigma}{2} \sum_{k \neq h} q_{ik} q_{ih}$$

where s parameterizes substitution intensity:

- $s > 0$: inputs are substitutes;
- $s < 0$: inputs are complements.
- $s = 0$ (our case) inputs are **independent**

iii) (Linearity) for any i there exist a vector $B_{i,\varepsilon} \in \mathbb{R}^{d_i}$ and a matrix $B_i \in \mathbb{R}^{d_i \times d_i}$ such that for all $(p_i, \varepsilon_i) \in \mathcal{O}_i^p \times \mathcal{O}_i^\varepsilon$:

$$\begin{pmatrix} S_i \\ D_i \end{pmatrix} = B_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} + B_{i,\varepsilon}\varepsilon_i \tag{10}$$

iv) (feasibility) for all i, $0 \in \mathcal{O}_i^\varepsilon$; moreover denote as $c(\mathcal{O}_i^p) = \{x \in \mathbb{R}_+^{|L|} \mid x_i = p_i\}$ the *cylinder* on $\mathcal{O}_i^p$. Then the realized price if $\varepsilon = 0$, call it p_0, belongs to $\times_{i\in I} c(\mathcal{O}_i^p)$;

Note that i implies that B_i is positive definite for all i, because it is the Jacobian of the schedule with respect to $(p_i^{out}, -p_i^{in})$.

This game is in principle very complex to solve, being defined on an infinite-dimensional space. In practice however, things are simpler, because a standard feature of competition in supply schedules, both in the finance and IO flavors, is that the best reply problem can be transformed from an ex-ante optimization over supply functions can be substituted by an ex-post optimization over input and output prices, as functions of the realizations of the parameter ε_i. In this way the best reply computation is reduced to an optimization over prices as in a monopoly problem. The crucial complication that the input-output dimension adds to e.g. Malamud and Rostek (2017) is the way the residual demand is computed. In an oligopoly without input-output dimension, as in Example 1 the residual demand is the portion of the final demand that is not met by competitors. In our context this remains true, but players will have to take into account how the prices of the goods each other firm is trading affects the balance of trades in the rest of the network. Let us first define it in general.

Definition 4.2 (Residual demand)**.** Given a profile of linear symmetric schedules $(((S_{i\alpha})_\alpha)_i, (D_{i\alpha})_\alpha)_i$, define the *residual demand*, and the *residual supply* of sector i as, respectively:

$$D_{ik}^r(p_k^*, p_i, \varepsilon_k, \varepsilon_i) = \underbrace{n_k D_{ki}((p_k^{out}, p_k^{in})^*, \varepsilon_k)}_{\text{demand from sector } k} - \underbrace{(n_i - 1)S_{ki}(p_i^{out}, p_i^{in}, \varepsilon_i)}_{\text{supply by competitors}}$$

$$S_{ij}^r(p_j^*, p_i, \varepsilon_j, \varepsilon_i) = \underbrace{n_j S_{ij}((p_j^{out}, p_j^{in})^*, \varepsilon_j)}_{\text{supply from sector } j} - \underbrace{(n_i - 1)D_i(p_i^{out}, p_i^{in}, \varepsilon_i)}_{\text{demand by competitors}} \quad \forall j : j \to i$$

where all the prices p^* of goods that are not bought not sold from firms in sector i are expressed as functions of p_i *solving the market clearing conditions* 5.

Example 2 (Line network)**.** The easiest setting in which to understand the mechanics of the residual demand is a line network, as illustrated in Figure 6.

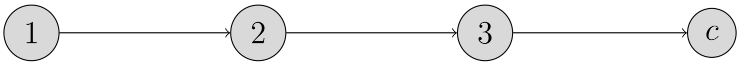


Figure 6: A line production network.

What is the residual demand (and supply) in this setting? to understand this, consider a firm in sector 1 that needs to compute its best reply to the schedules chosen by all others. (Details can be found in the Proof of Theorem 1). The demand curve faced by a firm in sector 1 is:

$$\underbrace{n_2 D_2(p_2^*, p_1, \varepsilon_2)}_{\text{Direct demand from sector 2}} \quad - \underbrace{(n_1 - 1) S_1(p_1, \varepsilon_1)}_{\text{Supply of competitors}}$$

for different choices of a supply function $S_{1\alpha}$, different prices p_1 would realize, as functions of the realizations of ε_2. For the best-responding firm, it is equivalent then to simply choose the price p_1 it would prefer for any given ε_1, and then the function $S_{1\alpha}$ can be backed up from these choices. But naturally also p_2^* is determined in equilibrium, and this has to be taken into account when optimizing. In particular, the market clearing conditions for sector 2:

$$n_2 S_{2\alpha}(p_2, p_1, \varepsilon_2) = D(p_2) + \varepsilon_c$$

define implicitly p_2 as a function of p_1 and the shocks. This allows to internalize in the price setting problem of firm 1 the impact that the variation in p_1 is going to have on p_2, for given supply and demand schedules chosen by other players. The same reasoning holds for the supply function. If we assume that all other players are using *linear* supply and demand schedules $S_1(p_1, \varepsilon_1) = B_1(p_1 - \varepsilon_1)$, $D_2(p_2, p_1, \varepsilon_2) = B_2(p_2 - p_1 - \varepsilon_2)$ we get the following expressions for the residual demands:

$$D_1^r = \frac{n_2 B_2}{B_c + n_2 B_2} (A_c + \varepsilon_c - B_c p_1) - (n_1 - 1) B_1 (p_1 - \varepsilon_1) \tag{11}$$

$$S_2^r = \frac{n_2 B_2}{B_c + n_2 B_2} (A_c + \varepsilon_c - B_c p_1) - (n_2 - 1) B_2 (p_2 - p_1 - \varepsilon_2) \tag{12}$$

$$D_2^r = A_c + \varepsilon_c - B_c p_2 - (n_2 - 1) B_2 (p_2 - p_1 - \varepsilon_2) \tag{13}$$

which clarifies how, even if each firms acts "locally" choosing its own input and output prices, actually the problem depends from the parameters of the whole economy.

The input-output matrix Residual demand and supply are the curves against which each firm will be optimizing when choosing its preferred input and output prices. It is natural therefore that they embed the information about relative market power. The key way through which the structure of the economy (i.e. the network) impacts these functions is via the dependence of the prices p^* on the input and output prices of i. To understand this, consider the market clearing equations.

The market clearing equations 5 define a system:

$$\begin{aligned} S_{il} &= D_{il} \quad \forall i, l \in N,\ i \to l \\ S_{i,c} &= D_{i,c} \quad \forall i \in N,\ i \to c \end{aligned} \tag{14}$$

If all other firms are using symmetric linear schedules with coefficients $(B_i)_i$, then this is a linear system, because all equations are linear in prices. We care about the solution of the system, so the ordering of the equations does not really matter. Let us rewrite the linear supply and demand schedules in a block form as:

$$\begin{pmatrix} S_i \\ D_i \end{pmatrix} = B_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} + \varepsilon_i B_{i,\varepsilon} = \begin{pmatrix} BS_i^{out} & BS_i^{in} \\ BD_i^{out} & BD_i^{in} \end{pmatrix} \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} + \varepsilon_i B_{i,\varepsilon}$$

(In case the sector employs only labor for production the matrices BD are empty).

So we can rewrite the system 14 as:

$$n_l BS_{l,i\cdot}^{out} p_l^{out} - n_l BD_{l,i\cdot}^{in} p_l^{in} - n_i BD_{i,l\cdot}^{out} p_i^{out} + n_i BD_{i,l\cdot}^{in} p_i^{in} = 0 \quad \forall i, l \in N,\ i \to l \tag{15}$$

$$n_i BS_{i,c\cdot}^{out} p_i^{out} - n_i BD_{i,c\cdot}^{in} p_i^{in} + B_{i,c} p_{i,c} = A_{i,c} \quad \forall i \in N,\ i \to c \tag{16}$$

$$BS_{l,i\cdot}^{out} p_i^{out} - BD_{l,i\cdot}^{in} p_i^{in} \geq 0 \quad \forall i \in N,\ i \to l, \quad p \geq 0 \tag{17}$$

To clarify the structure note that the market clearing equation for link $l \to i$ involves all prices of trades in which sectors l and i are involved.

Definition 4.3 (Market clearing coefficient matrix)**.** The *Market clearing coefficient matrix* corresponding to a profile of symmetric linear supply and demand schedules $(S_i, D_i)_i$ is the matrix M of dimension $|E| \times |E|$, where E is the set of edges of the production network $\mathcal{G}$, such that the market clearing system 14 in matrix form is:

$$Mp = \boldsymbol{A} + M_\varepsilon \varepsilon \tag{18}$$

$$B_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} + \varepsilon_i B_{i,\varepsilon} \geq 0 \tag{19}$$

$$p \geq 0 \tag{20}$$

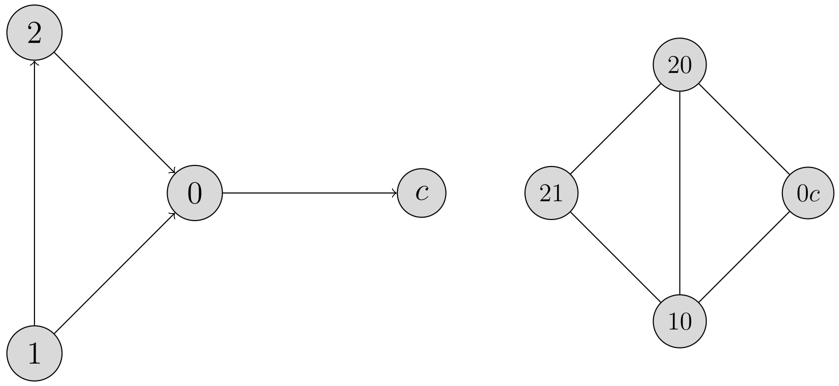


Figure 7: (left) A simple production network: c represents the consumer demand, while the other numbers index the sectors. (Right) The *line graph* of the network nearby.

The vector of constants $\boldsymbol{A}$ is zero but for the entries corresponding to links to the consumer (that have value $\boldsymbol{A}_{ci} = A_{ci}$).

This matrix M is the fundamental source of network information in this setting: it is a matrix indexed on the set of *links* of the network (which correspond to prices and equations in 5), that has a zero whenever two links do not share a node, and p is a vector that stacks all the prices. To have an example, consider the graph in Figure 7 case in which sector 0 has two suppliers: 1 and 2, and 1 itself supplies 2. If the profile of coefficients is $(B_i)_i$, the matrix M (when rows and columns are appropriately ordered) is:

$$\begin{array}{c} \\ (1\to 0) \\ (2\to 0) \\ (1\to 2) \\ (0\to c) \end{array} \begin{array}{c} \begin{array}{cccc} p_{10} & p_{20} & p_{12} & p_{0c} \end{array} \\ \begin{pmatrix} B_{1,11}+B_{0,22} & B_{0,23} & B_{1,12} & -B_{0,12} \\ B_{0,32} & B_{0,33}+B_{2,11} & -B_{2,12} & -B_{0,13} \\ B_{1,21} & -B_{2,21} & B_{1,22}+B_{2,22} & 0 \\ -B_{0,21} & -B_{0,31} & 0 & B_c+B_{0,11} \end{pmatrix} \end{array}$$

We can see that the only zero is in correspondence of the pair of links $(0, c)$ and $(1, 2)$ which indeed do not share a node.

In network-theoretic language this is the (weighted and signed) adjacency matrix of the *line graph* of the input-output network $\mathcal{G}$. That is the adjacency matrix of the network that has as nodes the link of $\mathcal{G}$ and such that two nodes share a link if and only if the corresponding links in $\mathcal{G}$ have a common sector. Note that this graph is *undirected*, which has the important implication that if all the coefficient matrices B_i are symmetric then also the matrix M is.

To obtain the residual demand, the linear system 14 can be partially solved to yield p^*_{-i} - the vector of all the prices of transactions in which sector i is not directly involved as a function of p_i:

$$p^*_{-i} = M_{-i}^{-1}(-M_{C_i} p_i + c_{-i})$$

where A_{-i} refers to all the rows of matrix A that do not involve links entering or exiting from node i, and M_{C_i} is the i-th column of M. This can be substituted in the supply and demand functions of suppliers and customers of i to yield the expression in the next proposition.

Proposition 2. *If all firms in all sectors $j \neq i$ are using linear supply and demand schedules with symmetric positive definite coefficients $(B_j)_j$, the residual supply and demand for sector i can be written as:*

$$\begin{pmatrix} -D_i^r \\ S_i^r \end{pmatrix} = \tilde{c}_i + ((n_i - 1)B_i + \Lambda_i^{-1}) \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix}$$

for all p_i such that this is positive[11].

Moreover, Λ_i is symmetric positive definite and equal to the matrix $[M_i^{-1}]_i$, where:

- *M_i is the matrix obtained by M by setting B_i to 0;*
- *if A is a matrix indexed by edges, $[A]_i$ is the submatrix of A relative to all the links that are either entering or exiting i.*

The coefficient Λ_i can be thought as a (sector level) *price impact*[12]: the slope coefficients of the (inverse) supply and demand schedules, describing what effect on prices firms in sector i can have. It is a measure of market power: the larger the price impact, the larger the rents firms in that sector can extract from the market.

[11] For all p_i outside of this set, the function can be extended uniquely as in Federgruen and Hu (2016). To do it, first solve the following Linear Complementarity Problem:

$$\tilde{c}_i + ((n_i - 1)B_i + \Lambda_i^{-1})(\tilde{p}_i - t) \geq 0 \quad t \geq 0 \quad t'(\tilde{c}_i + ((n_i - 1)B_i + \Lambda_i^{-1})(\tilde{p}_i - t)) = 0$$

that has a unique solution because $(n_i - 1)B_i + \Lambda_i^{-1}$ is positive definite. Then define

$$\begin{pmatrix} D_i^r \\ S_i^r \end{pmatrix} = \tilde{c}_i + ((n_i - 1)B_i + \Lambda_i^{-1})(\tilde{p}_i - t)$$

[12] Using a financial terminology. It is also the reason for the notation: from Kyle (1989) it is common to denote Λ the price impact of traders.

Now we can state the theorem. Define the *perfect competition matrix* for sector i as

$$C_i = \begin{pmatrix} \omega_i'\omega_i I^{out} & u_i^{out}\omega_i' \\ \omega_i(u_{out})_i' & d_i^{out} I^{in} \end{pmatrix}$$

Appendix **??** shows that this is the matrix of coefficients of demands and supplies chosen by a firm that takes prices as given.

Theorem 1. *1. If there are at least 3 firms involved in any exchange a non-trivial linear and symmetric Supply and Demand Function equilibrium exists, for a generic set of weights* Ω*;*

2. there is a subset of links where trade is nonzero, call it E_0*. The equilibrium is isomorphic to the equilibrium on the network* $G_0 = (N, E_0)$*.*

3. The equilibrium coefficients (referred to the network G_0*)* $(B_i)_i$ *can be found by iteration of the best reply map, starting:*

- *"from above": the perfect competition matrix* C_i*;*
- *"from below": any sufficiently small (in 2-norm) initial guess.*

4. The coefficients $(B_i)_i$ *can be written as*

$$B_i = \begin{pmatrix} \tilde{u}_i'\tilde{B}_i\tilde{u}_i & \tilde{u}_i'\tilde{B}_i \\ \tilde{B}_i\tilde{u}_i & \tilde{B}_i \end{pmatrix}$$

for a symmetric positive definite $\tilde{B}_i$*. The equations that characterize them are:*

$$\tilde{B}_i = \left([C_i^{-1}]_{-1} + ((n_i - 1)\tilde{B}_i + \overline{\Lambda}_i)\right)^{-1} \tag{21}$$

where $\overline{\Lambda}_i$ *is the* constrained price impact*:*

$$\overline{\Lambda}_i = [\Lambda_i^{-1}]_{-1} - \frac{1}{\tilde{u}_i'\Lambda_i^{-1}\tilde{u}_i}[\Lambda_i^{-1}\tilde{u}_i\tilde{u}_i'\Lambda_i^{-1}]_{-1}$$

and

$$B_{i,\varepsilon} = -\left(J_i - \frac{1}{k_i}J_i\tilde{u}_i\tilde{u}_i'J_i\right)\begin{pmatrix} \mathbf{0} \\ u_i^{in} \end{pmatrix}$$

The trivial equilibrium in which every supply and demand function are constantly 0, and so no unilateral deviation yields any profit because there would not be trade anyway, is always present[13]. The condition that at least three firms participate in any exchange is analogous to what happens in financial models (Malamud and Rostek (2017)). Part 1) guarantees that if at least 3 firms participate in the exchange there exists a non-trivial one.

Part 2) will be important for the numerical solution of the model, as discussed in Section 6.

The *constrained price impact* that appears in equation 21 is the matrix that represents the projection on the space of vectors that satisfy the technology constraint $\sum_k q_{ki} = \sum_j \omega_{ij} q_{ij}$. It is thus the necessary adaptation of the concept to an input-output setting: the technology constraint restricts the degrees of freedom that firms have in impacting the market price.

The expression for the best reply highlights the role of the price impact. If $\Lambda = 0$ then $B_i = C_i$ and the outcome is perfect competition. Moreover, we can see that also if $n_i \to \infty$ also the model predicts the perfect competition outcome, as it naturally is.

The proof consists of the following steps:

a) The optimization in the functions space can be reduced to an "ex-post" optimization over prices;

b) The best reply to a linear schedule is unique and still linear;

c) Each best reply coefficient B_i is increasing in all the other coefficients in the positive semidefinite ordering;

d) The iteration of best replies converges. In particular, since it converges from below, it converges to a non-trivial equilibrium.

The steps *a*) and *b*) follow the same principles of Klemperer and Meyer (1989) and Malamud and Rostek (2017). The main difference is that we have the input-output dimension, embodied by the residual demand. Step d) is necessary because despite step c) the positive semidefinite ordering does not make any matrix space a lattice[14], hence the standard theory of monotone comparative statics cannot be applied. Steps c) and d) can be considered generalizations to this setting of the results in Malamud and Rostek (2017). All proofs are in the Appendix.

To complete the section, I state the following corollaries.

[13] This is a feature of the particular technology used, in which labor is a perfect complement to intermediate inputs. In principle if this were not the case a firm producing some final good might find profitable to deviate from the no-trade equilibrium using some labor to sell to the consumers. This would break our assumption on the technology and the linearity of the equilibrium though.

[14] For a counterexample, see Malamud and Rostek (2017)

The first concerns a partial uniqueness result. Consider sector i, and consider *given* a profile of coefficients of firms in other sectors, that is, consider the sector level price impact Λ_i as given.

Corollary 4.1. If we consider the *sector-level* game played just by firms in sector i, this has a unique linear symmetric equilibrium.

The next corollary shows that in an interior equilibrium we do not need to worry about exit of firms: profits are never negative.

Corollary 4.2. In equilibrium, if quantities are nonnegative, profits can be expressed as:

$$\pi_i = \left((p_i^{out})^*, -(p_i^{in})^*\right) \left(B_i - \frac{1}{2} V_i' C_i V_i\right) \begin{pmatrix} (p_i^{out})^* \\ -(p_i^{in})^* \end{pmatrix}$$

where $V_i = \tilde{C}_i B_i + \frac{1}{k_i} \tilde{u}_i \tilde{u}_i' \Lambda_i^{-1} (I_i - \tilde{C}_i B_i)$

In particular since $B_i - \frac{1}{2} V_i' C_i V_i$ is positive semidefinite, so profits are always nonnegative in equilibrium.

5 The role of the network

In this section I discuss through some examples how the structure of the production network impacts market power. To understand how the network structure reflects on equilibrium outcomes, the crucial object to look at is, not surprisingly, the price impact. The simplest case is the one of trees.

A *tree* is a network in which either each node has just one input or each node has just one output. In particular, an example of a regular tree is illustrated in Figure 8. In the case of a tree it is possible to understand very precisely how the price impacts behave. Let us begin with a lemma characterizing the supply and demand coefficients.

Lemma 1. If the network is a tree, in a linear symmetric S&D equilibrium the coefficient matrices have the form:

$$B_i = \begin{pmatrix} \omega_i' \tilde{B}_i \omega_i & \omega_i' \tilde{B}_i \\ \tilde{B}_i \omega_i & \tilde{B}_i \end{pmatrix}$$

where $\tilde{B}_i$ is a matrix with positive diagonal entries and negative off-diagonal entries.

The particular form of the coefficients imply that the market clearing coefficient matrix M has a familiar Leontief form, as illustrated in the next Proposition.

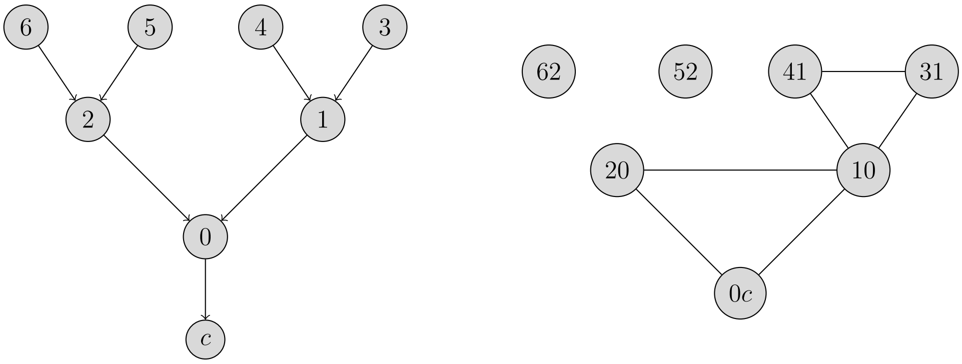


Figure 8: Left: A production network shaped as a regular tree. c represents the consumer demand, while the other numbers index the sectors. Right: the reduced line graph with respect to sector 2.

Proposition 3. *If the network is a tree, the entries of M^{-1} count the number of* undirected *paths connecting the nodes in the network. In particular:*

$$M^{-1} = (I - L)^{-1} D^{-1}$$

where L is a normalized adjacency matrix of the line graph $\mathcal{L}(\mathcal{G})$.

Now with the help of Proposition 2, we can understand how the price impact relates to the network. Indeed, according to Proposition 2, to obtain the price impact of say node 2 first we have to eliminate the links of the line graph connecting input and output links of 2. This is equivalent to building the line graph of the *reduced network* $\mathcal{G}_{-2}$, from which we removed the node 2. Since this is a tree now we have two separate subnetworks. These are illustrated in Figure 8 (right). Then, by a reasoning similar to Proposition 3 above, the entries of the matrix Λ_2 count the number of weighted paths between input and outputs of 2. But since in the reduced network input and output links are disconnected, the matrix is diagonal, and can be partitioned into:

$$\Lambda_i = \begin{pmatrix} \tilde{D}_i^{-1} & \mathbf{0} \\ \mathbf{0} & \tilde{S}_i^{-1} \end{pmatrix}$$

where $\tilde{D}_i^{-1}$ is the (weighted) number of self loops of the output link in $\mathcal{L}(\mathcal{G})$, and $\tilde{S}_i^{-1}$ is the matrix with on the diagonal the number of self loops of the input links in $\mathcal{L}(\mathcal{G})$.

Figure 9 illustrates the network intuition between the decomposition of Λ. It is very similar to the line network: the more upstream the sector is, the larger the portion of the network in which the "self-loops" have to be calculated. Hence the more elastic the demand it is facing. This is because a larger portion of the network is involved in the determination of the demand, and each price variation will distribute on a larger fraction of firms. The intuition is precisely the reverse for the supply coefficients, represented in Figure 10

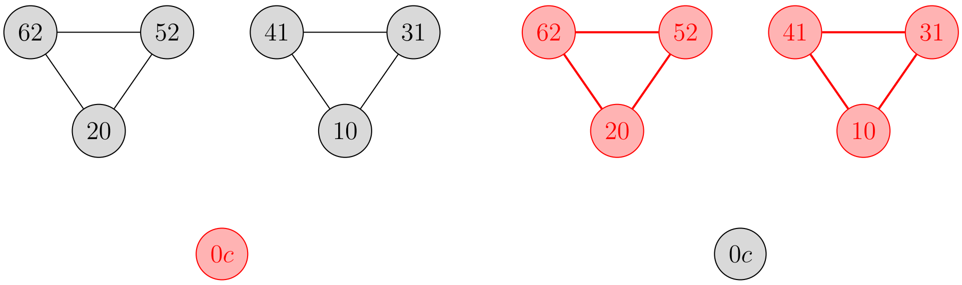


Figure 9: The relevant subnetworks of the line graph $\mathcal{L}(\mathcal{G})$ for the calculation of the price impact of sector 2. Left: output, right: inputs.

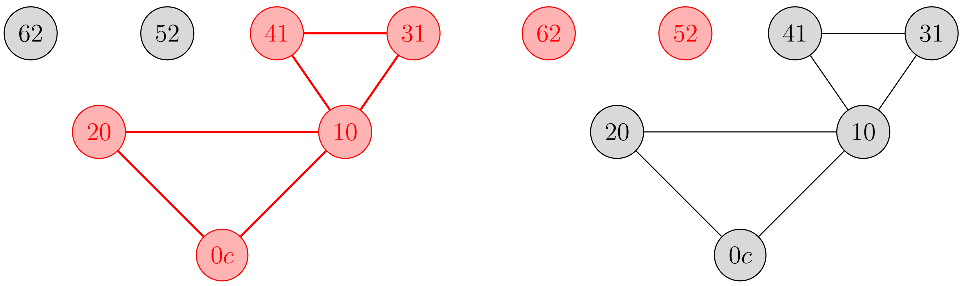


Figure 10: The relevant subnetworks for the calculation of the price impact for sector 0. Left: output, right: inputs.

Similar reasonings are at work for other networks, with the difference that in general inputs and outputs are not independent in the reduced network. Consider for example the network in Figure 7. What is the price impact of sector 2? In Figure 11 is represented the reduced network. Since now input and output links of sector 2 are connected, this means that Λ_2 is not diagonal anymore.

5.1 Market power and mergers

I turn now to the question of which sector is more relevant to merger policy. First, I show in an example that revenues are not a sufficient statistics for sector importance in this setting.

Example 3 (**Revenues are not a sufficient statistics**). Consider a tree oriented differently than in Section 2, as in Figure 12, consider the case in

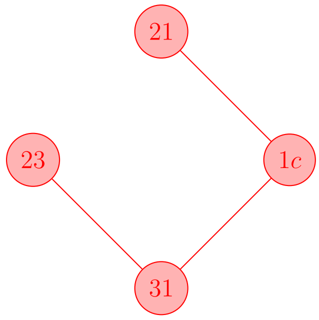


Figure 11: The subnetwork of the line graph in Figure 7 for the calculation of the price impact .

which the technology is such that $\omega_1 = \omega_2 = \omega_{01} = \omega_{02}$ and all sectors have the same number of firms. In this case parameters are balanced such that *all sectors have the same revenues*. Yet, as in the figure nearby, the welfare loss from mergers is very different in sector 1 and sector 0: it is almost double in sector 0! This shows that a policy maker ignoring the network dimension but focusing only on revenues would choose poorly the sector on which to focus on.

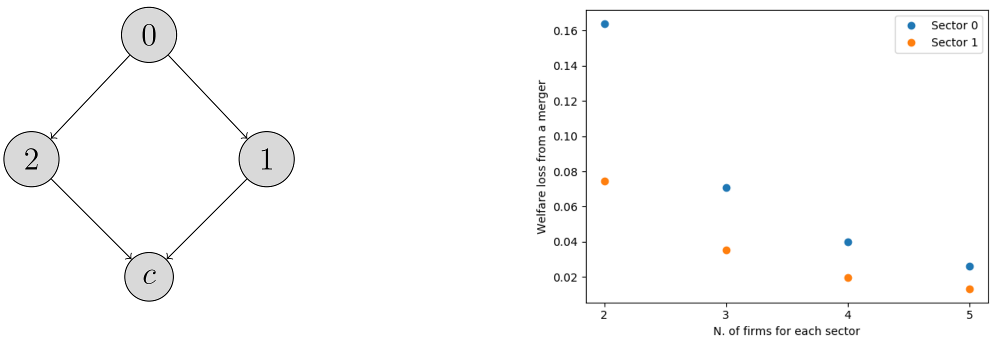


Figure 12: On the Left the network considered in the example, on the Right the welfare loss from a merger for different initial numbers of firms.

The following Theorem makes it precise the connection between mergers and price impact. The equilibrium price impact after the merger (which in this context is simply a reduction in the number of firms) cannot be larger in the sector with smaller price impact. Hence the price impact Λ gives a measure of relative damage of mergers in different sectors.

Theorem 2. *Assume i and j have the same number of customers and suppliers. If $\Lambda_i^* \geq \Lambda_j^*$, then a merger in sector j cannot increase equilibrium price impact more than a merger in sector i, that is:*

$$\Lambda_j^*(n_j - 1) - \Lambda_j^*(n_j) \not\geq \Lambda_i^*(n_i - 1) - \Lambda_i^*(n_i)$$

In particular, if $\Lambda_j^*(n_j - 1) - \Lambda_j^*(n_j)$ and $\Lambda_i^*(n_i - 1) - \Lambda_i^*(n_i)$ are comparable, the equilibrium price impact increases more in sector i than in sector j.

In the special case in which there is just one consumer good, we can obtain a result more direclty connected to welfare.

Corollary 5.1. If $\Lambda_i^*(n_i) \geq \Lambda_j^*(n_j)$ and $\Lambda_j^*(n_j - 1) - \Lambda_j^*(n_j)$ and $\Lambda_i^*(n_i - 1) - \Lambda_i^*(n_i)$ are comparable, the equilibrium consumer price increases more in sector i than in sector j:

$$p_c^*(n_j - 1) - p_c^*(n_j) \not\geq p_c^*(n_i - 1) - p_c^*(n_i)$$

To explore an example, let us focus on the regular tree of Figure 8, and let us assume $\omega_{ij} = \frac{1}{d_i^{in}}$, so that all inputs have the same relative weight in production. Because this choice of technology, this setting allows particularly sharp predictions. This is because, given the symmetry of the problem, all the sectors in the tree will produce the same quantity of output q_i, *no matter the mode of competition.* Hence focusing on this case it is useful can abstract from reallocation and size effects. In the Appendix **??** I show that in this case under perfect competition profits are identical for all firms.

The results for the S&D equilibrium are numerically calculated in Figure 13. It turns out that the equilibrium price impacts are increasing as one moves toward the root of the tree, hence the Corollary above applies in its most useful form. The sector which is the most essential for connecting the whole network is able to extract a larger surplus, and the other are progressively less important the farther upstream one goes.

The importance for the regulator follows the same pattern. Figure 14 shows that the welfare loss from a merger that brings the number of firms from 2 to 1 is larger in sector 0, and smaller the more we move upstream.

Example 4 (S&D Equilibrium vs Sequential competition)**.** The line network, as in Example 1 is a good setting to gain intuition because sharper results can be obtained. In particular, we can characterize the differences of the sequential competition models with the Supply and Demand Function Equilibrium. The next proposition makes precise the intuition expressed in Section 2 that the perceived elasticity of demand tends to decrease as we get closer to the first mover. This has implications for the markups and markdowns.

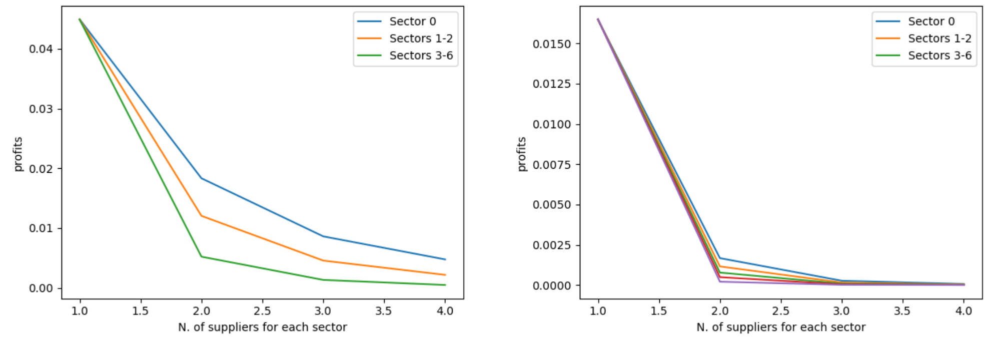


Figure 13: Profits for regular trees of height 2 (Left), and 4 (Right), for different numbers of suppliers. Sector 0 is always making the larger profit (except with 1 supplier, which is the case of the line). The number of firms is set to 2 in each sector.

Proposition 4. *Consider the supply chain illustrated in Figure 4. Assume the consumer has a concave and differentiable demand function, firms have an identical, concave and differentiable production function f, and there is the same number of firms in each sector. Moreover, assume that at each step of the backward induction the inverse demand remains concave*[15].

Then in the output-setting game in quantities firms in sector 1 have larger markup; in the input-setting game in quantities firms in sector 2 have larger markup.

To better compare with the S& D Equilibrium, consider the case in which firms have the same quadratic technology introduced in 3.2.

What is the analogous of the markup in this bilateral setting? to understand this, let us write the problem of the firm in its general form (as in section 3.2):

$$\max p_i D_i^r(p_i) - p_{i-1} S_i^r - \frac{1}{2} z_i^2 \tag{22}$$

subject to:

$$D_i^r(p_i, p_{i-1}) = z_i \tag{23}$$
$$S_i^r(p_i, p_{i-1}) = z_i \tag{24}$$

this form is naturally redundant in the case of this simple network. Now define μ_i, the *marginal value of inputs* as the Lagrange multiplier relative to the second constraint, and λ_i, the *marginal value of output*, as the Lagrange

[15]These conditions are for example satisfied if the technology and utility are quadratic.

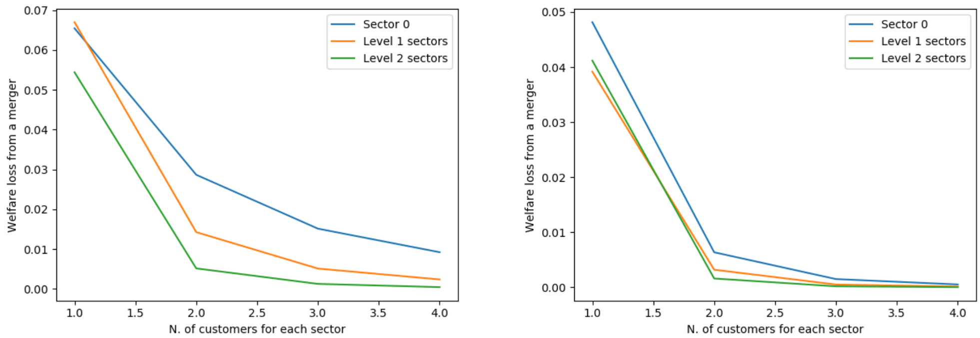


Figure 14: Welfare loss from a merger that brings the number of firms from 2 to 1 in different sectors, for different number of suppliers. Left: tree of height 2, Right: tree of height 4.

multiplier relative to the first constraint. Then we can define simultaneously a *markup* and a *markdown*:

$$M_i = p_i - \lambda_i \quad m_i = \mu_i - p_{i-1} \tag{25}$$

which are both zero under perfect competition. The next propostion characterizes their behavior.

Proposition 5. *If $n_1 = n_2$, sector 2 has a larger markup, while sector 1 has a larger markdown.*

This clarifies that the behavior of elasticities in sequential models does not disappear: but here the bilateral nature of the game makes it possible to *both* effects to manifest. How do they balance? To understand who is earning a larger surplus, let us focus on the profit.

The profit of firms in the symmetric S&D equilibrium can be rewritten as:

$$\pi_i = (M_i + m_i)q_i + \frac{1}{2}q_i^2$$

which makes the intuition transparent: remembering that q_i is constant, the profit in excess of the common component depends on the magnitude of the *sum* of markup and markdown. The next proposition

Proposition 6. *In the symmetric Supply and Demand Function Equilibrium for the Supply Chain, the sector with larger profit is the sector with the smallest number of firms. In particular if $n_1 = n_2$ then all sectors have the same profit.*

So, contrary to the sequential competition models, in a S&D equilibrium in a supply chain no one is privileged with respect to others. This follows from the fact that no sector can substitute away from others, they are all essential to produce the consumer good. This allows to shed light on the sequential competition shortcomings: when market power is bilateral one needs to take into account *simultaneously* markup and markdown. When doing so, the paradox disappears and the basic intuition is recovered.

6 Numerical implementation

The solution by iteration of best reply makes the model numerically tractable for medium sized networks. The main bottleneck is the inversion of the market clearing matrix M, which being a matrix links-by-links, tends to be huge, especially if the network is not very sparse. An application of the Matrix inversion lemma (or Woodbury formula, see Horn and Johnson (2012)) allows to invert the full matrix just once, and then at each step update the inverse by just inverting a small matrix, of size equal to the degree of the involved sector. The gain in this process is especially large when the network is sparse because then the matrices to be inverted are small. The algorithm for solving the model numerically is:

1. initialize all the matrices $B_{i,0}$ as C_i;
2. initialize all relative errors of all nodes to some large number, e.g. 1;
3. start from some node $\hat{\imath}$. Compute the best reply, inverting the matrix M, and save the inverse.
4. choose the node that has the maximum relative error E_i. Compute its best reply. In doing so, update the inverse of the matrix M using the Matrix Inversion Lemma;
5. Repeat 4 until all E_i are smaller than a threshold (I use 0.01).

In Figure 15 I show the computation time to reach the equilibrium for Erdos-Renyi random graphs of 200 nodes, of different densities.

7 Conclusion

I build a model of trade among firms as a game in supply and demand function, which allows to study the problem of how the exogenously given network of firm interactions contributes to determine market power. In the

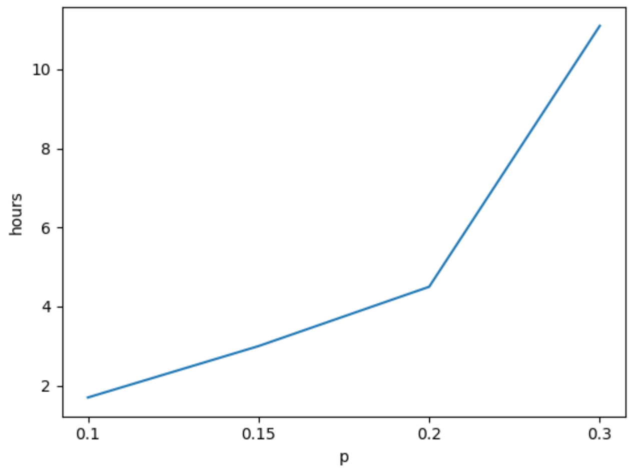


Figure 15: Time of model solution for ER random graphs, 200 nodes, average degree $=200p$. Iteration stopped when maximum percentage error $< 0.1\%$

case of a tree network, it is possible to connect the endogenous matrix of price impacts to the intuitive notion of Bonacich centrality. I conjecture that the connection is general. Though Bonacich centrality appears often in input-output economics, in this model I show that not only the size of a firm depends on its position, but also its ability to affect prices. The size of a firm (measured e.g. by revenues) will depend on centrality even under perfet competition, as is well known. Here I am introducing another margin: besides begin large, central firms will have more ability to affect market prices. This is a result that can be of interest in the line of research that explores misallocation and its welfare effects.

The results in Section 6 show that it is actually possible to use this model in networks of a realistic dimension. A full exploration of the insights that can be obtained from real data is an interesting area to develop further. As I have shown through some examples, the model can in principle be used to assess the market impact of mergers as a function of the position in the network, which might be of interest for antitrust authorities.

References

Acemoglu, D. and A. Tahbaz-Salehi (2020, July). Firms, failures, and fluctuations: The macroeconomics of supply chain disruptions. Working Paper 27565, National Bureau of Economic Research.

Baqaee, D. R. (2018). Cascading failures in production networks. *Econometrica 86*(5), 1819–1838.

Baqaee, D. R. and E. Farhi (2017). Productivity and misallocation in general equilibrium. Technical report, National Bureau of Economic Research.

Benassy, J.-P. (1988). The objective demand curve in general equilibrium with price makers. *The Economic Journal 98*(390), 37–49.

Bimpikis, K., S. Ehsani, and R. İlkılıç (2019). Cournot competition in networked markets. *Management Science 65*(6), 2467–2481.

Bonanno, G. (1990). General equilibrium theory with imperfect competition 1. *Journal of economic surveys 4*(4), 297–328.

Carvalho, V., M. Elliott, and J. Spray (2020). Supply chain bottlenecks during a pandemic.

De Bruyne, K., G. Magerman, J. Van Hove, et al. (2019). Pecking order and core-periphery in international trade. Technical report, ULB–Universite Libre de Bruxelles.

Delbono, F. and L. Lambertini (2018). Choosing roles under supply function competition. *Energy Economics 71*, 83–88.

Dierker, H. and B. Grodal (1986). Non-existence of cournot-walras equilibrium in a general equilibrium model with two oligopolists,"contributions to mathematical economics. *Honor of Gerard Debreu"(W. Hildebrand and A. MassColell, Eds.), North-Holland, Amsterdam.*

Federgruen, A. and M. Hu (2016). Sequential multiproduct price competition in supply chain networks. *Operations Research 64*(1), 135–149.

Fleiner, T., R. Jagadeesan, Z. Jankó, and A. Teytelboym (2019). Trading networks with frictions. *Econometrica 87*(5), 1633–1661.

Gabszewicz, J. J. and J.-P. Vial (1972). Oligopoly "a la cournot" in a general equilibrium analysis. *Journal of economic theory 4*(3), 381–400.

Gale, D. and H. Nikaido (1965). The jacobian matrix and global univalence of mappings. *Mathematische Annalen 159*(2), 81–93.

Grassi, B. (2017). Io in io: Competition and volatility in input-output networks. *Unpublished Manuscript, Bocconi University.*

Green, R. J. and D. M. Newbery (1992). Competition in the british electricity spot market. *Journal of political economy 100*(5), 929–953.

Hatfield, J. W., S. D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp (2013). Stability and competitive equilibrium in trading networks. *Journal of Political Economy 121*(5), 966–1005.

Hendricks, K. and R. P. McAfee (2010). A theory of bilateral oligopoly. *Economic Inquiry 48*(2), 391–414.

Hinnosaar, T. (2019). Price setting on a network. *Available at SSRN 3172236*.

Horn, R. A. and C. R. Johnson (2012). *Matrix analysis*. Cambridge university press.

Huremovic, K. and F. Vega-Redondo (2016). Production networks.

Kikkawa, K., G. Magerman, and E. Dhyne (2019). Imperfect competition in firm-to-firm trade. *Available at SSRN 3389836*.

Klemperer, P. D. and M. A. Meyer (1989). Supply function equilibria in oligopoly under uncertainty. *Econometrica: Journal of the Econometric Society*, 1243–1277.

Kotowski, M. H. and C. M. Leister (2019). Trading networks and equilibrium intermediation.

Kyle, A. S. (1989). Informed speculation with imperfect competition. *The Review of Economic Studies 56*(3), 317–355.

Malamud, S. and M. Rostek (2017). Decentralized exchange. *American Economic Review 107*(11), 3320–62.

Marschak, T. and R. Selten (2012). *General equilibrium with price-making firms*, Volume 91. Springer Science & Business Media.

Nikaido, H. (2015). *Monopolistic Competition and Effective Demand.(PSME-6)*. Princeton University Press.

Ordover, J. A., G. Saloner, and S. C. Salop (1990). Equilibrium vertical foreclosure. *The American Economic Review*, 127–142.

Salinger, M. A. (1988). Vertical mergers and market foreclosure. *The Quarterly Journal of Economics 103*(2), 345–356.

Tintelnot, F., A. K. Kikkawa, M. Mogstad, and E. Dhyne (2018). Trade and domestic production networks. Technical report, National Bureau of Economic Research.

Weretka, M. (2011). Endogenous market power. *Journal of Economic Theory 146*(6), 2281–2306.